Additional Results of the Capacity Bounds in Cellular Systems

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Abstract

In this paper, we extend our previously published results on the capacity of cellular systems to include the effects of shadow fading and power control. These studies investigate how the reuse factor can be improved given the knowledge of mobiles' locations. We use square-root power control, which we prove is the optimal power control strategy.

1 Introduction

A large body of research has been published on the performance of cellular systems (e.g., [1] - [6]). Most of these papers present and analyze schemes in which the granularity of mobiles' location is the cell size. In this work, we assume that more information about mobiles' location is available to the channel allocation algorithm and show the reduction in the number of channels required to support some mobiles' population.

Cellular systems rely on either Fixed Channel Allocation (FCA) schemes or Dynamic Channel Allocation (DCA) schemes. In FCA, channels are permanently assigned to cells, usually in some repetitious pattern to ensure some level of co-channel interference. In DCA, a pool of channels is shared among cells, providing the trunking gain. The actual channel allocation in DCA may be based on some predetermined allocation rules (e.g., an allocated channel precludes reallocation within some fixed radius) or on interference measurement on channels, to determine whether a channel is allocable to a specific mobile (e.g., [7]).

In our work, we define the maximum packing algorithm as an algorithm that requires minimum number of channels (MNC) to support a given number of preallocated mobiles in each cell. MNC is a static bound and may be used for comparison of practical channel assignment schemes.

In general, the MNC bound will vary with the location of the mobiles. However, our results show that with randomly-located mobiles (the random locations drawn from the uniform distribution), the variation in the bound for different realizations of the random locations is relatively small. Therefore, we use Monte Carlo simulation to determine the bound for a few realizations and use these results to obtain an approximate bound for most cases of randomly located users.

In practice, we may not need to assign channels to all mobiles, but can block some percentage of arriving calls. However, because in any practical system the blocking probability needs to be extremely small ($\approx 2\%$ for current analog systems), the MNC including or excluding blocking will not change significantly.[1]

As mentioned, our results are static. However, if the time variations of the mobile density or the probability distribution function of mobile density is known, one may readily determine the distribution of the required number of channels.

This paper is organized as follows: In the next section, we briefly describe our model and the procedures used to obtain MNC. In section 3, we summarize previously published results, showing the effects of the different parameters on the MNC. Additional data showing the effects of shadow fading and power control on the MNC is discussed in section 4. Finally, in section 5, we summarize and conclude the study. In the Appendix, we prove that the square-root power control strategy used in obtaining additional data is, in fact, the optimal power-control strategy.

2 Background Information

The following model is used throughout our work:

1. The cellular structure is composed of 33 internal cells, with a base-station located in the middle of each cell.
2. Except when implicitly stated, fixed number of mobiles is randomly distributed within each cell.
3. We assume that there is only one co-channel interference (i.e., the *single interferer* assumption). This assumption is justified later in the paper.
4. The up link and the down link channels are paired.
5. Our model accounts for Rayleigh fading of moving (i.e., non-stationary) mobiles only and for propagation loss with Shadow fading (see section 4).
6. We assume TDMA and FDMA channel assignment.

[1] This claim is suggested by the Figures 2 and 3.

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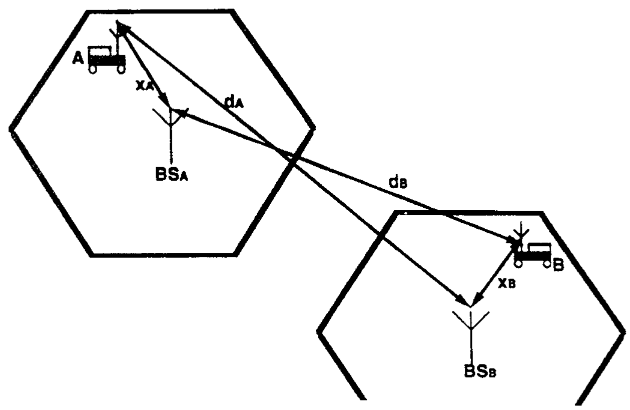

Figure 1: Interference distances

7. Base-stations and mobiles are equipped with omni-directional antennas.

The parameters used in the study are:

- n - number of mobiles in each cell
- N - total number of mobiles in the system
- α - the *interference radius* - the minimum required ratio between the distance from the base-station to the interferer and the distance from the base-station to the mobile
- P_B - the power transmitted by a base-station (we assume all the base-stations transmit the same power level)
- P_M - the power transmitted by a mobile (we assume all the mobiles transmit the same power level)
- S - the signal's received power
- I - the level of received interfering signal
- r - the propagation loss exponent (where applicable, we assume $r = 3.8$)
- N_c = MNC - the number of channels required to accommodate the n users under some given conditions
- x - the distance of a mobile to its base-station
- d - the distance of a mobile to other than its own base-station

The study evaluates the minimum required number of channels (referred to here also as colors) when some knowledge about mobiles' location is given, such that some maximum interference conditions are satisfied. The interference conditions refer to the acceptable level of interference, so that two mobiles can be assigned the same channel.

Interference Conditions

We assume that there is some minimum SIR[2], termed SIR_{min}, required for sufficient system operation. We define α to be:

$$\alpha = \sqrt[r]{\text{SIR}_{min}}. \quad (1)$$

As shown in [8] (refer to Figure 1 for distance definitions), the set of conditions to be satisfied to ensure SIR_{min} is:

$$d_A \geq \alpha \cdot x_A; \; d_B \geq \alpha \cdot x_A; \; d_B \geq \alpha \cdot x_B; \; d_A \geq \alpha \cdot x_B \quad (2)$$

[2] Signal-to-Interference Ratio

Evaluating the MNC

First, we draw the location of the n mobiles for each cell from the uniform distribution. Then a matrix of dimension $N \times N$ is constructed (referred to here as the ***compatibility matrix***); i.e., each mobile in the whole system is evaluated with each other mobile and based on the interference conditions (2) it is determined whether the two mobiles can be assigned the same channel – i.e., two mobiles that satisfy the conditions (2) are declared compatible; otherwise, they are incompatible.

A graph is then composed, where each mobile in the system corresponds to a vertex in the graph. Two vertices are interconnected by an edge in the graph, if and only if the two mobiles represented by the two vertices are "incompatible"; i.e., they cannot be assigned the same channel. A set of graph coloring algorithms is then employed to find the minimum number of colors to color the vertices in the graph, such that no two vertices interconnected by an edge are colored in the same color. Each color corresponds then to a channel assigned to the mobiles that the vertices colored in that color represent. Thus, the MNC equals the number of colors. Note that the number of colors needed is at least the size of the maximum clique[3] in the graph.

For a small number of vertices, finding the number of colors is a relatively easy task. However, to obtain any meaningful statistics, instances with large number of mobiles need to be evaluated. Thus, since the graph coloring and the maximum clique problem are NP-complete problems ([9]), finding the minimum number of colors can be a complex task, as can be computing the lower bound provided by the maximum clique size. In this work, we thus use an algorithm suggested by [10] that, while not finding the exact minimum number of colors required, is remarkably effective at finding good upper bounds on that number for the types of graphs we create. We then complement our upper bounds with lower bounds determined by heuristics for the maximum clique problem, thus pinning the precise value of the MNC[4] within a narrow range, and indeed often determining it exactly.

Other approaches to cope with the large complexity of the problem have been proposed. In particular, simulated annealing and neural networks[5] were used ([11]). Our algorithm is significantly faster than the simulated annealing and neural net approaches and finds bounds that are typically just as good, or, in the latter case significantly better.

Note that, practically speaking, while knowing the location of each mobile may not be possible, all that is necessary for maximal packing is the knowledge of the signal levels of each mobile's signal at all the base stations and

[3] A *clique* is a subgraph in which every two vertices are interconnected by an edge.

[4] That is required to support communication to/from all the mobiles with SIR greater than SIR_{min}.

[5] Such schemes usually suffer from the uncertainty whether the obtained solution is, in fact, optimal and what is the deviation from the optimal solutions. Our "brute force" scheme presented here provides either an 'exact' solution or tight bounds within which the optimal result exists.

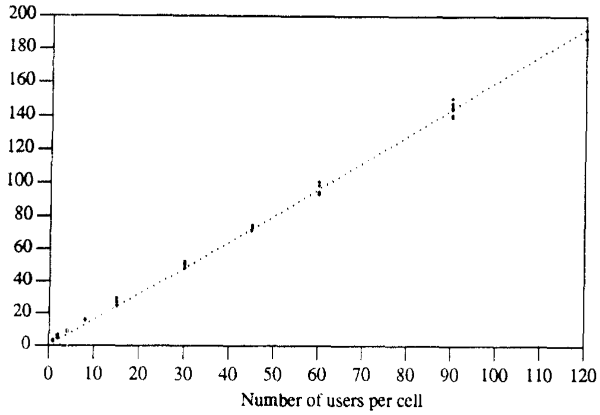

Figure 2: Number of channels as a function of mobile density

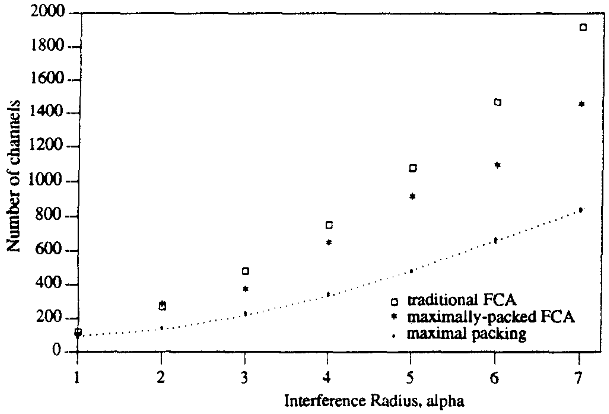

Figure 3: Number of channels as a function of the interference radius

the knowledge of each base station signal at all the mobiles. With such knowledge, the resulting SIR for a specific channel allocation can be calculated. This, in turn, can be used to determine the compatibility of the mobiles under such an allocation.

3 Bounds on the Number of Channels

In this section, we show the effect of the various system parameters on the minimum number of channels, N_c.

- The effect of the number of mobiles per cell is shown in Figure 2, demonstrating that the number of channels increases linearly with the mobile density, measured here in mobiles-per-cell. The results shown in this figure were obtained with $\alpha = 2.0$, which corresponds to reuse pattern of 3. The slope of the curve in the figure is about 1.6. Thus, for $\alpha = 2$, the reduction in the number of channels offered by maximal packing vs. fixed reuse pattern (e.g.,

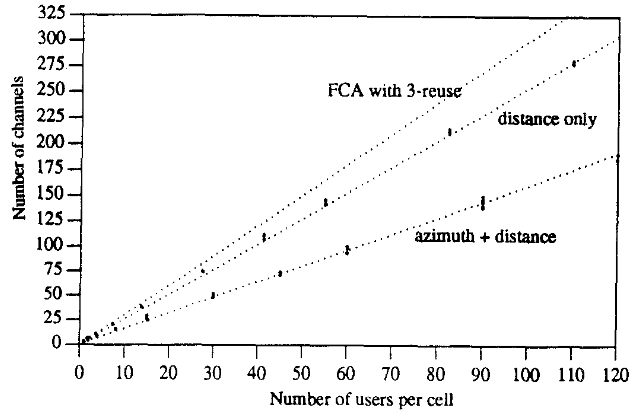

Figure 4: Number of channels as a function of mobile density, when the azimuth is not known

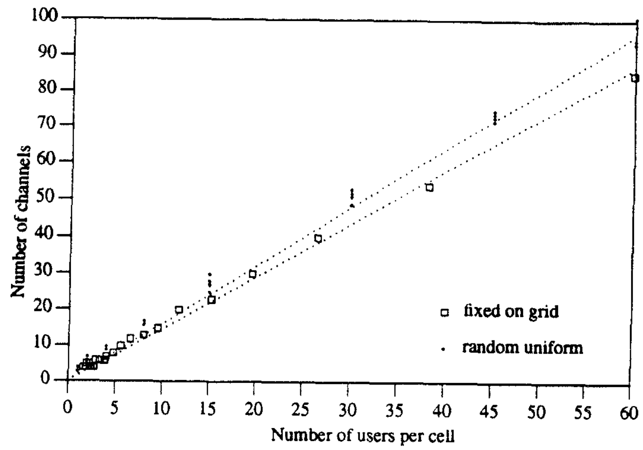

Figure 5: Number of channels as a function of mobile density for uniformly distributed mobiles

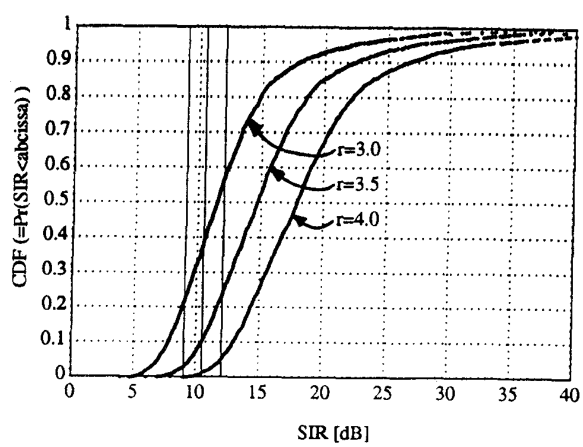

Figure 6: Cumulative probability distribution function of the mobiles' SIR

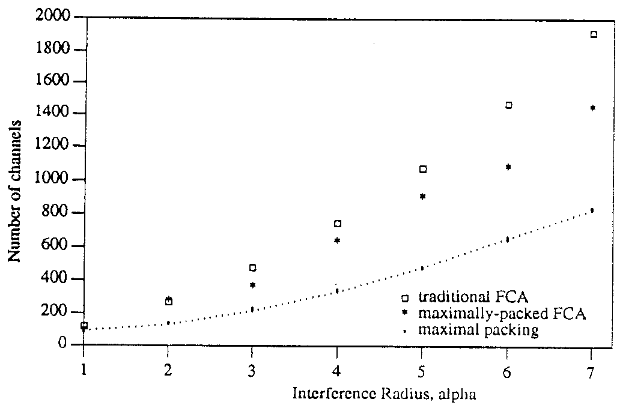

Figure 7: Number of required channels as a function of cell subdivisions

FCA) is approximately a factor of 2.

• The effect of the interference radius, shown in Figure 3, demonstrates the dependence of the number of required channels on the interference radius, α, for the maximal packing scheme. The results were obtained for 90 mobiles per cell. This figure also compares the relative reduction in the number of channels of the maximal packing scheme with the maximally-packed FCA scheme.[6] The gain of the maximal packing algorithm compared to maximally-packed FCA is approximately constant at about 40% over the range of $\alpha = 2$ to $\alpha = 7$.

• The effect of direction (azimuth) ignorance is shown in Figure 4. Results in this figure were obtained with $\alpha = 2.0$. As an example, for 90 mobiles per cell, the number of required channels with no knowledge of direction is reduced by 24%, compared with the reduction of 52% when the direction is also known. In a practical system, it may suffice to measure the received powers between the mobiles and their associated base-station instead of the distances.[7]

• The effect of mobile distribution within the cell is shown in Figure 5, where the improvement of fixed location (on a uniform grid) over random distribution is demonstrated. This improvement, which is about 15%, is due to the elimination of "hot spots," that occur in the random distribution. The results were obtained with $\alpha = 2.0$.

• In [8], it was shown that **full** power control has no effect on the number of channels. However, square-root power control, as discussed in the next session, can considerably affect the MNC.

• The "single-interferer" assumption was shown to affect the results only marginally. Figure 6 depicts the probability distribution function of the mobiles' SIR, when the allocation was done relying on the single-interferer assumption. The figure indicates that only about 7% of the users have their SIR below the designed-for SIR_{min} (assuming $r = 3.8$ and α=2.0). Consequently, the corresponding error in the minimum number of channels due to the single-interferer assumption may be considered negligible.

• The effect of subdividing cells into subcells is shown in Figure 7. The granularity of mobiles' location is now limited to the size of the subcell. As shown, subdivision into approximately 20 subcells suffices for the MNC to be 'close' to the limiting case where the location of the users is perfectly known (i.e., granularity is zero, or the number of subdivisions is infinite).

[6] The maximally-packed FCA scheme assumes that the channel assignment is fixed to the cells, but is not necessarily repetitious.

[7] When power control is performed, additional information on the transmitted power needs to be conveyed to the receiver site.

4 Effects of Shadow Fading and Power Control

In this section, we present some results showing the effect of shadow fading and power control on the performance of the maximal packing scheme.

Shadow fading is modeled as a random variable (Θ) of log-normal distribution with σ=8 dB and with propagation exponent r=3.8. (In the case where there is no shadow fading, the propagation loss exponent remains at r=3.8.) It is assumed that the shadow fading is transmission direction independent; i.e., its value for the up- and the down-link between two points is the same.

The power control strategy is to partially compensate for the propagation with the shadow fading. We use square-root power control, which is proven in the Appendix to be optimal (see also [12]). The square-root power control effectively reduces the loss (with or without the shadow fading) to half of their values in dB.

The results are shown in Figure 8. The shadow fading affects the compatibility matrices, making them considerably less sparse. This causes larger range between the lower and the upper bounds, making it more difficult to draw definite conclusions.

Nevertheless, Figure 8 shows the improvement of the power control mechanism, both with and without the shadow fading. An interesting observation is that the shadow fading may actually reduce the number of required channels, especially for large α. This conclusion can be intuitively justified by observing that the shadow fading tends to more 'positively' affect the mobiles that are further away from the base-station than the mobiles closer to the base-station. And since, because of the uniform distribution, there are more mobiles further away than closer to the base-station, the overall effect is an increase in the compatibility among mobiles. The 'positive' effect mentioned above refers to the fact that the shadow fading either increases or decreases the effective distance of the mobile from the base-station. Thus two incompatible mobiles may become compatible by decreasing their distance from

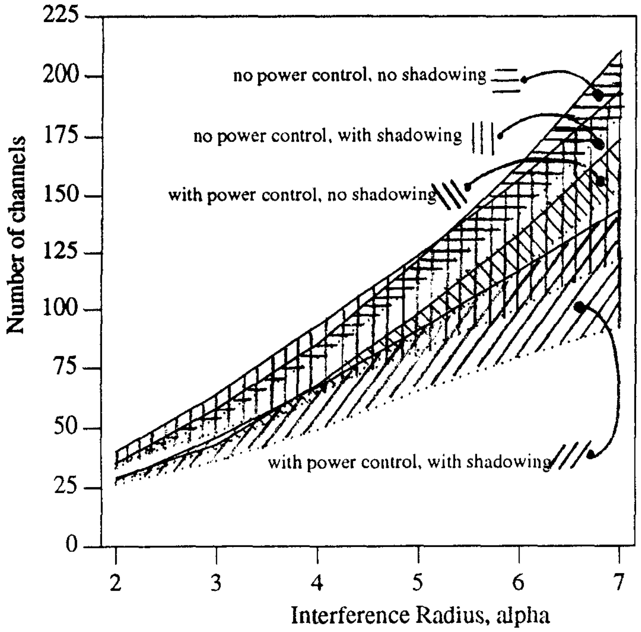

Figure 8: The effects of shadow fading and power control

the base-station. (Increasing the distance of two compatible mobiles may render them incompatible. Yet, because of the smaller number of mobiles closer to the base-station, this tendency is a less favorable one.)

This effect is of particular importance for large α-s, since small changes in the effective distance of a mobile to a base-station may be enough to convert a larger number of incompatible to compatible mobiles.[8]

5 Concluding Remarks

In this paper, we have investigated the performance of the maximum packing algorithm for cellular structure. Our results are innovative in the sense that the channel assignments are done based on the knowledge (or limited knowledge) of the mobiles' locations. Thus, the cellular structure does not, by itself, limit the reuse of channels. An example of such a system is one in which the mobiles (and the base stations) measure the amount of interference to determine the available channels. The assignments are then done based on these measurement results.

Our investigation used heuristics for graph-coloring and clique-finding to obtain bounds on the minimum number of channels needed in each situation. The heuristics were chosen both for their speed and for the quality of bounds they yielded, and in these respects seemed to significantly outperform other methods such as simulated annealing and neural nets.

The results of this study indicate that the maximal packing scheme presented here can reduce the number of channels by nearly a factor of 2, for an interference radius of 2.0. When only the distance (or power level) between each mobile and its nearest base station is known, the reduction in the number of channels is somewhat modest, 25% for interference radius of 2.0. More uniform mobile distribution reduces the number of required channels, since mobile clustering tends to reduce the channel reuse. We found about a 15% difference in the number of channels between the random (uniform) and fixed (on square grid) cases (for interference radius of 2.0).

We have proven the optimality of the square-root power control strategy and investigated its performance, showing that the square-root power control reduces the number of channels by about 20% for $\alpha = 2$ to more than 30% for $\alpha = 7$. Furthermore, we have justified that the single-interferer assumption used throughout our study corresponds to a small 7% error (for r=3.5) in the number of channels. Additionally, our results are also valid (with some minimal error) when a small amount of blocking is allowed in the system.

Finally, we have investigated the effect of the shadow fading, surprisingly concluding that shadow fading may, in fact, reduce the required number of channels, especially for large interference radii. The observed degradation due to the shadow fading at small α-s is relatively minor.

The schemes shown here may be of particular interest to PCN, where the user density may be considerable and efficient spectrum reuse may be crucial.

6 Acknowledgement

The authors would like to thank Mona Singh and David Applegate for helping in implementing the graph-coloring and the clique finding algorithms that we have used in this work.

7 Appendix

In the Appendix, we prove the optimality of the square-root power control strategy with and without shadow fading. More precisely, we prove that when the propagation loss is compensated by adding (in dB) a fraction of the loss and fading, the optimal coefficient for such compensation equals 0.5 .

The definitions of distances are given in Figure 1. Additionally, we denote by $P_F(A)$, where F and A equals M_1, M_2, B_1, or B_2, the power from mobile or base F received at mobile or base A. Θ_{FA} represents the shadow fading (random variable) between F and A, where F and

[8] It is as if the changes in the effective distance due to the shadow fading are amplified by α.

A can be either a mobile or a base-station. Finally, κ is the propagation loss compensation constant.

We start by evaluating the power from the mobiles received at **their** base-stations including the compensation factor[9]:

$$P_{M_1}(B_1) = P_M \Theta_{M_1B_1} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa}, \quad (3)$$

$$P_{M_2}(B_2) = P_M \Theta_{M_2B_2} \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa}. \quad (4)$$

Similarly, we evaluate the power received at the base-stations from **their** mobiles after compensation:

$$P_{B_1}(M_1) = P_B \Theta_{M_1B_1} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa}, \quad (5)$$

$$P_{B_2}(M_2) = P_B \Theta_{M_2B_2} \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa}. \quad (6)$$

Now, we evaluate the interference level of mobiles on other than their base-stations and of base-stations on other than their mobiles. Note that in these equations, the compensation is based on the signal path[10], while the fading is based on the interfering path[11].

$$P_{M_1}(B_2) = P_M \Theta_{M_1B_2} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa}, \quad (7)$$

$$P_{M_2}(B_1) = P_M \Theta_{M_2B_1} \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa}, \quad (8)$$

$$P_{B_1}(M_2) = P_B \Theta_{M_2B_1} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa}, \quad (9)$$

$$P_{B_2}(M_1) = P_B \Theta_{M_1B_2} \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa}. \quad (10)$$

In order for the two mobiles to be compatible, the power levels must satisfy the following set of conditions:

$$P_{M_1}(B_1) \geq \alpha^r \cdot P_{M_2}(B_1) \quad (11)$$

$$P_{M_2}(B_2) \geq \alpha^r \cdot P_{M_1}(B_2) \quad (12)$$

$$P_{B_1}(M_1) \geq \alpha^r \cdot P_{B_2}(M_1) \quad (13)$$

$$P_{B_2}(M_2) \geq \alpha^r \cdot P_{B_1}(M_2) \quad (14)$$

Substituting equations (3) and (8) into the inequality (11) results in:

$$P_M \Theta_{M_1B_1} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa} \geq \alpha^r P_M \Theta_{M_2B_1} \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa}, \quad (15)$$

which can be further rearranged to give:

$$\alpha^r \Theta_{M_2B_1} \leq (\Theta_{M_1B_1})^{1-\kappa} (\Theta_{M_2B_2})^{\kappa}. \quad (16)$$

Similarly, substituting equations (6) and (9) into the inequality (14) results in:

$$P_B \Theta_{M_2B_2} x_2^r \left(\frac{1}{\Theta_{M_2B_2}}\right)^{\kappa} \geq \alpha^r P_B \Theta_{M_2B_1} \left(\frac{1}{\Theta_{M_1B_1}}\right)^{\kappa}, \quad (17)$$

which after rearrangement yields:

$$\alpha^r \Theta_{M_2B_1} \leq (\Theta_{M_1B_1})^{\kappa} (\Theta_{M_2B_2})^{1-\kappa}. \quad (18)$$

In order to maximize the probability of the two mobiles being compatible, we need to maximize the minimum of the righthand sides of inequalities (16) and (18) with respect to κ. I.e., we will attempt to find:

$$\max_{\kappa} \left\{ \min \left[(\Theta_{M_1B_1})^{\kappa} \cdot (\Theta_{M_2B_2})^{1-\kappa} \right], \left[(\Theta_{M_1B_1})^{1-\kappa} \cdot (\Theta_{M_2B_2})^{\kappa} \right] \right\}. \quad (19)$$

Since in (19) one term is an increasing while the other is a decreasing function of κ, the maximum of the minimum of the two terms occur when the two terms are equal. Thus:

$$(\Theta_{M_1B_1})^{\kappa} \cdot (\Theta_{M_2B_2})^{1-\kappa} = (\Theta_{M_1B_1})^{1-\kappa} \cdot (\Theta_{M_2B_2})^{\kappa}, \quad (20)$$

which, after some manipulations and for arbitrary $\Theta_{M_1B_1}$, and $\Theta_{M_2B_2}$, yields:

$$\kappa = \frac{1}{2}. \quad (21)$$

Proceeding with the inequalities (12) and (13) in a similar manner yields the same optimal value for κ. The above proof assumed the presence of shadow fading. Substituting $\Theta_{XY} = x_i^{-r}$ for all X and Y, where x_i is the distance between X and Y, proves the optimality of $\kappa = 0.5$ for the no shadow fading case.

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[9] Note that, in practice, the compensation is done at the transmitting end by increasing the transmitted power based on the measurements at the receiving end.

[10] Path between base-station and its mobiles or between mobile and its base-station

[11] E.g., path between a base-station and other than its mobiles

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