Analysis of Hybrid Selection/Maximal-Ratio Combining in Rayleigh Fading

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Abstract—**The performance of a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system in a multipath-fading environment is analyzed. With H-S/MRC, L out of N diversity branches are selected and combined using maximal-ratio combining (MRC). This technique provides improved performance over L branch MRC when additional diversity is available, without requiring additional electronics and/or power. In particular, we consider independent Rayleigh fading on each diversity branch with equal signal-to-noise ratio averaged over the fading. We analyze this system using a "virtual branch" technique which results in a simple derivation and formula for the mean and the variance of the combiner output SNR for *any* L and N.**

I. Introduction

The CAPACITY of wireless systems in a multipath environment can be increased by diversity techniques [1]. Diversity gain is typically achieved by selection combining (SC) or maximal-ratio combining (MRC) [2]. SC is the simplest form of diversity combining whereby the received signal is selected from *one* out of N available diversity branches. In MRC, the received signals from *all* the diversity branches are weighted and combined to maximize the *instantaneous* signal-to-noise ratio (SNR) at the combiner output.

Though a high diversity order is possible in many situations, it may not be feasible to utilize all of the available branches. For example, a large order of antenna diversity may be obtained, especially at higher frequencies such as the PCS bands, using spatial separation and/or orthogonal polarizations. Even for a handset, the main limitation is typically not the handset size (which determines the maximum number of antenna elements) but rather the power consumption and cost of the RF electronics for each diversity branch [3]. For spread spectrum receivers operating in dense multipath environments, the number of resolvable paths (or diversity branches) increases as a function of transmission bandwidth [4], [5]. However, the available correlator resources limit the number of paths that can be utilized in a typical Rake combiner [5].

This has motivated studies of diversity combining techniques that process only a *subset* of the available diversity branches with limited resources (i.e., power, RF electronics), but achieve better performance than SC. This reduced-complexity combining system selects the L best branches (from N available diversity branches) and then combines the selected subset of branches based on a chosen criterion. Selecting the "best" branches can be accomplished by selecting the branches with the largest SNR or signal-plus-noise [6], [7]. The selected subset of branches can then be combined using equal gain combining or MRC [2], [8], [9]. Here, we consider the hybrid selection/maximal-ratio combining (H-S/MRC) diversity system which selects the L branches with largest SNR at each instant, and then combines these branches to maximize the SNR. We assume that instantaneous channel estimation using a scanning receiver across all possible diversity branches is feasible, such as with slow fading. However, H-S/MRC also offers improvement in fast fading conditions, and our results serve as an upper bound on the performance when perfect channel estimates are not available. H-S/MRC has been considered before as an efficient means to combat multipath fading [10], [11].

In this paper, we present a simple derivation of the SNR with H-S/MRC in Rayleigh fading using the "virtual branch" technique. In particular, we consider independent fading on each branch and assume that the SNR, averaged over the fading, is the same for all diversity branches. The average SNR gain of H-S/MRC is derived, since it is a commonly accepted performance measure of diversity systems [2]. In order to assess the effectiveness of H-S/MRC system in the presence of multipath, the variance of the combiner output is also derived. The power of the virtual branch technique is even more apparent in the simplicity of the derivation for the variance.

II. Diversity Combining Analysis

A. Virtual Branch Technique: The Key Idea

The analysis of H-S/MRC based on a chosen ordering of the branches at first appears to be complicated, since the SNR statistics of the ordered-branches are *not* independent. Even the *average* combiner output SNR calculation alone can require a lengthy derivation as seen in [11]. Here, we alleviate this problem by transforming the ordered-branch variables into a new set of independent

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identically distributed (i.i.d.) *virtual branches*, and expressing the ordered-branch SNR variables as a linear function of i.i.d. virtual branch SNR variables. The key advantage of this formulation is that it allows greater flexibility in the selection process of the ordered instantaneous SNR values, and permits the combiner output SNR to be expressed in terms of the i.i.d. virtual branch SNR variables. In this way, the derivation of the moments of the combiner output SNR is essentially reduced to the calculation of the moments of the linear combination of i.i.d. random variables.

In this framework, the *average* SNR of the combined output is obtained in a less complicated manner than the derivation given in [11]. Furthermore, the extension to the derivation of the combiner output SNR variance can be made more succinctly using this virtual branch technique. The well-known results for SC and MRC are shown to be special cases of our results.

B. General Theory

Let γ_i denote the instantaneous SNR of the $i^{\underline{\text{th}}}$ diversity branch defined by

$$\gamma_i \triangleq \frac{g_i^2}{N_{0i}}, \tag{1}$$

where g_i^2 is the instantaneous signal power and N_{0i} is the noise power spectral density of the $i^{\underline{\text{th}}}$ branch. We model the γ_i's as continuous random variables with probability density function (p.d.f.) $f_{\gamma_i}(x)$ and mean $\Gamma_i = \mathbb{E}\{\gamma_i\}$.

Let us first consider a general diversity-combining (GDC) system with the instantaneous output SNR of the form

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} a_i \gamma_{(i)}, \tag{2}$$

where $a_i \in \{0, 1\}$, $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \ldots > \gamma_{(N)}$, and N is the number of available diversity branches. It will be apparent later that several diversity combining schemes, including H-S/MRC, turn out to be special cases of (2). Note that the possibility of at least two equal $\gamma_{(i)}$ is excluded, since $\gamma_{(i)} \neq \gamma_{(j)}$ *almost surely* for continuous random variables γ_i.[1]

For a Rayleigh fading channel, the p.d.f. of the instantaneous branch SNR is given by

$$f_{\gamma_i}(x) = \begin{cases} \frac{1}{\Gamma_i} e^{-\frac{x}{\Gamma_i}}, & 0 < x < \infty \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

If the instantaneous branch SNR's γ_i are independent with the same average SNR, i.e., $\Gamma_i = \Gamma$ for $i = 1, \ldots, N$, then $f_{\gamma_i}(x) = f(x) \quad \forall i$. Denoting $\boldsymbol{\gamma}_{(N)} \triangleq (\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)})$, the joint p.d.f. of $\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)}$ is [14]

$$f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N}) = \begin{cases} N! \left(\frac{1}{\Gamma}\right)^N e^{-\frac{1}{\Gamma}\sum_{m=1}^{N}\gamma_{(m)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

[1]The notion of "almost sure" or "almost everywhere" is commonly used in probability theory [12], [13]. In our context, it can be stated mathematically as: if $\mathcal{N} = \{\gamma_{(i)} = \gamma_{(j)}\}$, then $\Pr\{\mathcal{N}\} = 0$.

The mean SNR of combiner output signal is given by

$$\begin{aligned} \Gamma_{\text{GDC}} &= \mathbb{E}\{\gamma_{\text{GDC}}\} \\ &= \int_0^\infty \int_0^{\gamma_{(1)}} \cdots \int_0^{\gamma_{(N-1)}} \sum_{i=1}^{N} a_i \gamma_{(i)} \\ &\quad \times f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N})\, d\gamma_{(N)} \cdots d\gamma_{(2)} d\gamma_{(1)}. \end{aligned} \tag{5}$$

Since the statistics of the ordered-branches are *no* longer independent, the evaluation of the mean SNR involves nested integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR of the ordered diversity branches into a new set of *virtual branch* instantaneous SNR's, V_i, using the following relation:

$$\gamma_{(i)} = \sum_{n=i}^{N} \frac{\Gamma}{n} V_n. \tag{6}$$

It can be verified that the instantaneous SNR's of the virtual branches are i.i.d. normalized exponential random variables with p.d.f.'s given by

$$f_{V_n}(v) = \begin{cases} e^{-v}, & 0 < v < \infty \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

The instantaneous SNR of the combiner output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{\text{GDC}} = \sum_{n=1}^{N} b_n V_n, \tag{8}$$

where

$$b_n = \frac{\Gamma}{n} \sum_{i=1}^{n} a_i. \tag{9}$$

Using the virtual branches, the derivation for the moments of the combiner output SNR essentially reduces to the calculation of the moments of the linear combination of i.i.d. random variables. Using the fact that normalized exponential random variables have unity mean ($\mathbb{E}\{V_n\} = 1$), the mean of the combiner output SNR can now be calculated as

$$\Gamma_{\text{GDC}} = \mathbb{E}\left\{\sum_{n=1}^{N} b_n V_n\right\} = \sum_{n=1}^{N} b_n. \tag{10}$$

Similarly, the variance of the combiner output SNR is

$$\sigma_{\text{GDC}}^2 = Var\left\{\sum_{n=1}^{N} b_n V_n\right\} = \sum_{n=1}^{N} b_n^2, \tag{11}$$

where the unit variance of the normalized exponential random variable ($Var\{V_n\} = 1$) is used. Note that the independence of the virtual branch variables plays a key role in simplifying the derivation of (10) and (11).

The average SNR gain of the diversity combining compared to a single branch system is a commonly accepted performance measure [2]. This quantity is calculated as

$$G_{\mathrm{GDC}} \triangleq 10\log_{10}\left\{\frac{\Gamma_{\mathrm{GDC}}}{\Gamma}\right\} = 10\log_{10}\left\{\frac{\sum_{n=1}^{N} b_n}{\Gamma}\right\}. \quad (12)$$

To assess the effectiveness of diversity combining in the presence of multipath, we also define the normalized standard deviation of the combiner output SNR as

$$\sigma_{\mathrm{n,GDC}} \triangleq 10\log_{10}\left\{\frac{\sqrt{\sigma^2_{\mathrm{GDC}}}}{\Gamma_{\mathrm{GDC}}}\right\} = 10\log_{10}\left\{\frac{\sqrt{\sum_{n=1}^{N} b_n^2}}{\sum_{n=1}^{N} b_n}\right\}. \quad (13)$$

C. Hybrid S/MRC Analysis

In this section, the general theory derived in Section II-B is used to evaluate the performance of H-S/MRC. The instantaneous output SNR of H-S/MRC is

$$\gamma_{\mathrm{S/MRC}} = \sum_{i=1}^{L} \gamma_{(i)}, \quad (14)$$

where $1 \leq L \leq N$. Note that $\gamma_{\mathrm{S/MRC}} = \gamma_{\mathrm{GDC}}$ with

$$a_i = \begin{cases} 1, & i = 1, \dots, L \\ 0, & \text{otherwise}. \end{cases} \quad (15)$$

In this case,

$$b_n = \begin{cases} \Gamma, & n \leq L \\ \Gamma \frac{L}{n}, & \text{otherwise}. \end{cases} \quad (16)$$

Substituting (16) into (10) and (11), the mean and the variance of the combiner output SNR can be easily obtained as

$$\Gamma_{\mathrm{S/MRC}} = L\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)\Gamma, \quad (17)$$

and

$$\sigma^2_{\mathrm{S/MRC}} = L\left(1 + L\sum_{n=L+1}^{N} \frac{1}{n^2}\right)\Gamma^2, \quad (18)$$

respectively. Therefore the average SNR gain of H-S/MRC in dB is

$$G_{\mathrm{S/MRC}} = 10\log_{10}\left\{L\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)\right\}, \quad (19)$$

and normalized standard deviation of the combiner output SNR in dB is

$$\sigma_{\mathrm{n,S/MRC}} = 10\log_{10}\left\{\frac{\sqrt{\left(1 + L\sum_{n=L+1}^{N} \frac{1}{n^2}\right)}}{\sqrt{L}\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)}\right\}. \quad (20)$$

D. Limiting Case 1: SC System

SC is the simplest form of diversity combining whereby the received signal from *one* of N diversity branches is selected [2]. The output SNR of SC is

$$\gamma_{\mathrm{SC}} = \max_i \{\gamma_i\}. \quad (21)$$

Note that $\gamma_{\mathrm{SC}} = \gamma_{\mathrm{GDC}}$ with $a_1 = 1$ and $a_i = 0$ for $i = 2, \dots, N$. In this case, $b_n = \frac{\Gamma}{n}$, and substituting this into (10) and (11), the mean and variance of the combiner output SNR for SC becomes

$$\Gamma_{\mathrm{SC}} = \Gamma\sum_{n=1}^{N} \frac{1}{n}, \quad (22)$$

and

$$\sigma^2_{\mathrm{SC}} = \Gamma^2\sum_{n=1}^{N} \frac{1}{n^2}, \quad (23)$$

respectively. Therefore, the average SNR gain of SC is

$$G_{\mathrm{SC}} = 10\log_{10}\left\{\sum_{n=1}^{N} \frac{1}{n}\right\}, \quad (24)$$

and normalized standard deviation of the combiner output SNR for SC is

$$\sigma_{\mathrm{n,SC}} = 10\log_{10}\left\{\frac{\sqrt{\sum_{n=1}^{N} \frac{1}{n^2}}}{\sum_{n=1}^{N} \frac{1}{n}}\right\}. \quad (25)$$

Alternatively, (22), (23), (24), and (25) can also be obtained from the H-S/MRC results (17), (18), (19), and (20) by setting $L = 1$. This should be expected since SC is a limiting case of H-S/MRC with $L = 1$. Note also that the limiting result given in (22) agrees with the well-known result of mean SNR for the selection diversity given by (5.2.8) of [2, page 316].

E. Limiting Case 2: MRC System

In MRC, the received signals from *all* diversity branches are weighted and combined to maximize the SNR at the combiner output. The output SNR of MRC is given by

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{N} \gamma_{(i)}. \quad (26)$$

Note that $\gamma_{\mathrm{MRC}} = \gamma_{\mathrm{GDC}}$ with $a_i = 1 \;\; \forall i$. In this case, $b_n = \Gamma$, and the mean and variance of the MRC output SNR becomes $\Gamma_{\mathrm{MRC}} = N\Gamma$ and $\sigma^2_{\mathrm{MRC}} = N\Gamma^2$, respectively. Therefore the average SNR gain of MRC is

$$G_{\mathrm{MRC}} = 10\log_{10}\{N\}, \quad (27)$$

and normalized standard deviation of the MRC is

$$\sigma_{\mathrm{n,MRC}} = 10\log_{10}\left\{\frac{1}{\sqrt{N}}\right\}. \quad (28)$$

Note again that the results for MRC given above may also be obtained from the H-S/MRC results given in (17), (18), (19), and (20) by setting $L = N$ since MRC is a limiting case of H-S/MRC with $L = N$. The mean SNR for MRC obtained by the limiting procedure agrees with the well-known result given by (5.2.16) of [2, page 319].

III. Numerical Examples

In this section, the results derived in the previous section for H-S/MRC are illustrated. The notation H-L/N is used to denote H-S/MRC that selects and combines L out of N branches with the largest SNR to maximize the combiner output SNR. Note that H-1/1 is a single branch receiver, H-1/N is SC, and, H-N/N is MRC with N branches.

Figure 1 shows the average SNR gain with H-L/N as a function of L for various N. The data points denoted by the "squares" represent the average SNR gain using H-1/N. The data points denoted by the "stars" represent the average SNR gain using H-L/L and serve as a lower bound for the average SNR gain of H-L/N. It can be seen that H-L/N provides average SNR gain over H-L/L, when additional diversity is available, without requiring additional electronics and/or power. Figure 2 shows the average SNR gain using H-L/N versus N for various L. The data points denoted by the "squares" and the "stars" represent the average SNR gain using H-1/N and H-N/N, respectively. Note that the curves for H-1/N and H-N/N serve respectively as a lower and upper bound for the average SNR gain of H-L/N.

In Fig. 3, we plot the normalized standard deviation of the H-L/N combiner output SNR as a function of L for various N. The normalized standard deviation of the combiner output SNR for H-1/N and H-L/L can be seen as the two limiting cases. The curve for H-L/L upper bounds the normalized standard deviation of the H-L/N combiner output SNR. Note that H-L/N provides reduction in normalized standard deviation of the combiner output SNR compared to H-L/L, when additional diversity is available, without requiring additional electronics and/or power. Figure 4 shows the normalized standard deviation of the H-L/N combiner output SNR versus N for various L. It can be clearly seen that the curves for H-1/N and H-N/N upper and lower bound, respectively, the normalized standard deviation of the H-L/N combiner output SNR.

IV. Conclusions

We derived the average SNR gain as well as the variance of the combiner output SNR of a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system in a multipath fading environment. In particular, we considered independent Rayleigh fading on each diversity branch with equal signal-to-noise ratios, averaged over the fading. We analyzed this system using a "virtual branch" technique which resulted in a simple derivation and formulas for *any* L and N. The key idea was to transform the dependent ordered-branch variables into a new set of i.i.d. *virtual branches*, and express the combiner output SNR as a linear combination of the i.i.d. virtual branch SNR variables. In this framework, the moments of the combined output SNR can be derived succinctly. These results allow easy analysis of the improved performance of H-S/MRC over L branch MRC. Numerical evaluation was made for limited ranges of L and N. The well-known results for SC and MRC were shown to be special cases of our results.

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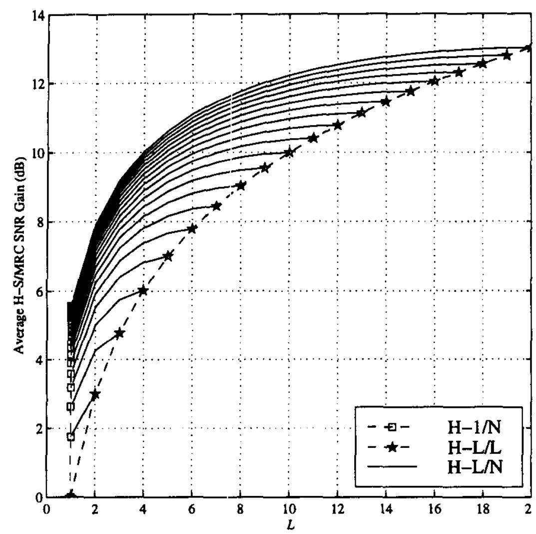

Fig. 1. The average SNR gain with H-L/N as a function of L for various N. The solid curves are parameterized by different N starting from the lowest curve with $N = 2$ and increase monotonically to the highest curve with $N = 20$.

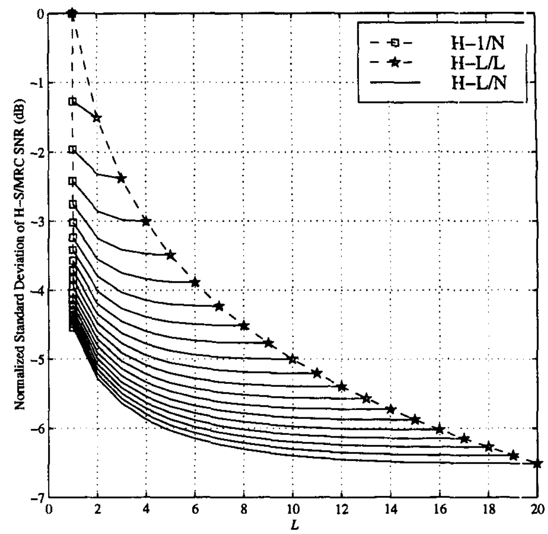

Fig. 3. The normalized standard deviation of the H-L/N combiner output SNR as a function of L for various N. The solid curves are parameterized by different N starting from the highest curve with $N = 2$ and decrease monotonically to the lowest curve with $N = 20$.

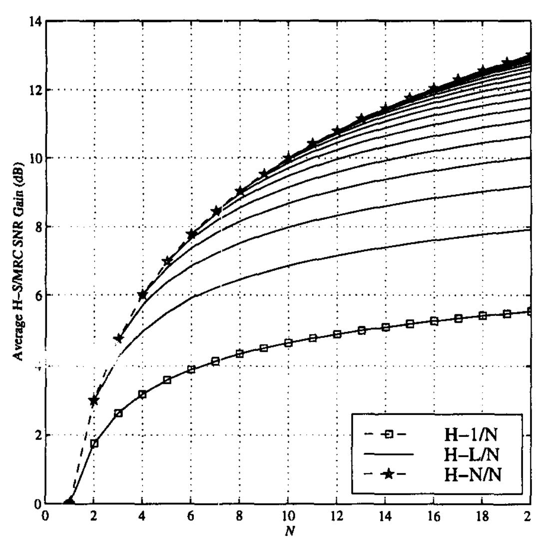

Fig. 2. The average SNR gain with H-L/N as a function of N for various L. The lowest solid curve represents $L = 1$ and increase monotonically to the highest solid curve with $L = 20$.

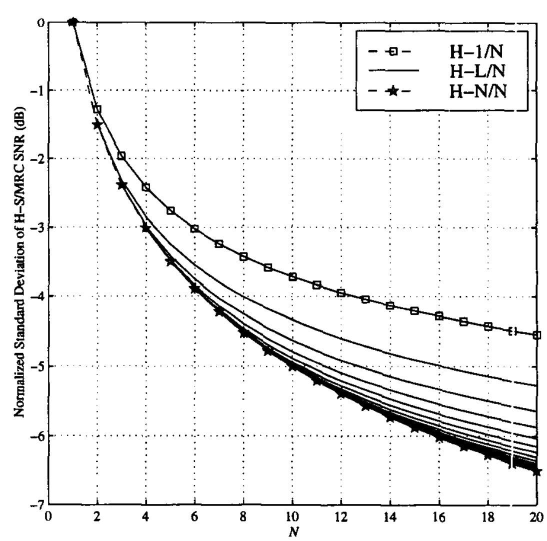

Fig. 4. The normalized standard deviation of the H-L/N combiner output SNR as a function of N for various L. The highest solid curve represents $L = 1$ and decrease monotonically to the lowest solid curve with $L = 20$.