Analysis of Hybrid Selection/Maximal-Ratio Combining of Diversity Branches with Unequal SNR in Rayleigh Fading

Moe Z. Win and Jack H. Winters

Wireless Systems Research Department, AT&T Labs - Research
100 Schulz Drive, Red Bank, NJ 07701-7033 USA
Tel: (732)-345-3144 Fax: (732)-345-3038
e-mail: {win, jhw}@research.att.com

Abstract—**The performance of a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system in a multipath-fading environment is analyzed. With H-S/MRC, L out of N diversity branches are selected and combined using maximal-ratio combining (MRC). This technique provides improved performance over L branch MRC when additional diversity is available, without requiring additional resources (i.e., power consumption, RF electronics). In particular, we consider independent Rayleigh fading on each diversity branch without assuming equal signal-to-noise ratio (SNR) averaged over the fading. We analyze this system using a "virtual branch" technique which results in a simple derivation and formula for the mean of the combiner output SNR for *any* L and N. We further obtain the symbol error probability (SEP) for coherent reception of M-ary phase-shift keying (MPSK) and quadrature amplitude modulation (MQAM).**

I. Introduction

THE CAPACITY of wireless systems in a multipath environment can be increased by diversity techniques [1]. Diversity gain is typically achieved by selection combining (SC) or maximal-ratio combining (MRC) [2]. SC is the simplest form of diversity combining whereby the received signal is selected from *one* out of N available diversity branches. In MRC, the received signals from *all* the diversity branches are weighted and combined to maximize the *instantaneous* signal-to-noise ratio (SNR) at the combiner output.

Though a high diversity order is possible in many situations, it may not be feasible to utilize all of the available branches. For example, a large order of antenna diversity may be obtained, especially at higher frequencies such as the PCS bands, using spatial separation and/or orthogonal polarizations. Even for a handset, the main limitation is typically not the handset size (which determines the maximum number of antenna elements) but rather the power consumption and cost of the RF electronics for each diversity branch [3]. For spread spectrum receivers operating in dense multipath environments, the number of resolvable paths (or diversity branches) increases as a function of transmission bandwidth [4], [5]. However, the available correlator resources limit the number of paths that can be utilized in a typical Rake combiner [5].

This has motivated studies of diversity combining techniques that process only a *subset* of the available diversity branches with limited resources (i.e., power consumption, RF electronics), but achieve better performance than SC. This reduced-complexity combining system selects the L best branches (from N available diversity branches) and then combines the selected subset of branches based on a chosen criterion. Selecting the "best" branches can be accomplished by selecting the branches with the largest SNR or signal-plus-noise [6], [7]. The selected subset of branches can then be combined using equal gain combining or MRC [2], [8], [9]. Here, we consider the hybrid selection/maximal-ratio combining (H-S/MRC) diversity system which selects the L branches with largest SNR at each instant, and then combines these branches to maximize the SNR. We assume that instantaneous channel estimation using a scanning receiver across all possible diversity branches is feasible, such as with slow fading. However, H-S/MRC also offers improvement in fast fading conditions, and our results serve as an upper bound on the performance when perfect channel estimates are not available.

Recently, H-S/MRC has been considered as an efficient means to combat multipath fading [10], [11], [12], [13]. The bit error rate (BER) performance of a H-S/MRC Rake receiver was analyzed in [10] for binary differential phase shift-keying modulation. In [10], the probability density function (p.d.f.) of the sum of the signals with the strongest path SNR's was obtained as a convolution of the p.d.f.'s of the strongest, the second strongest, ..., and the L^{th} strongest paths.[1] In [11], the BER performance of H-S/MRC with only $L = 2$ and $L = 3$ out of N branches was analyzed. The *average* SNR of H-S/MRC was derived in [12], where H-S/MRC is referred to as the "generalized diversity selection combining scheme." In [13], a "virtual branch" technique was introduced to succinctly derive the mean as well as the variance of the combiner output SNR of the H-S/MRC diversity system.

[1] We remark that, in general, p.d.f. of the random variable $\gamma = \sum_{i=1}^{L} \gamma_{(i)}$ is the convolution of the p.d.f.'s of $\gamma_{(i)}$ *only* if the random variables $\gamma_{(i)}$'s are *independent*.

0-7803-5565-2/99/$10.00 © 1999 IEEE

In this paper, we extend the results of [12] and [13], which assume independent Rayleigh fading with *equal* SNR averaged over the Rayleigh fading on each diversity branch, to the *unequal* branch SNR case. We analyze this system using a "virtual branch" technique, similar to that used in [13], which results in a simple derivation and formula for the mean of the combiner output SNR for *any* L and N. The key idea of the virtual branch technique as applied to the equal-SNR case [13] is to transform the dependent ordered-branch variables into a new set of *virtual branches* which are independent. When the average branch SNR's are not necessarily equal, we show here that the virtual branch technique can also be applied, but the virtual branches are now *conditionally* independent. We then express the combiner output SNR as a linear combination of the virtual branch SNR variables. We first obtain the mean of the combiner output SNR. We further obtain the symbol error probability (SEP) for coherent reception of M-ary phase-shift keying (MPSK) and quadrature amplitude modulation (MQAM). The power of the virtual branch technique is apparent in the simplicity of the derivations throughout the paper.

II. Diversity Combining Analysis

A. Virtual Branch Technique: The Key Idea

The analysis of H-S/MRC based on a chosen ordering of the branches at first appears to be complicated, since the SNR statistics of the ordered-branches are *not* independent. Even the *average* combiner output SNR calculation alone can require a lengthy derivation as seen in [12]. Here, we alleviate this problem by transforming the ordered-branch variables into a new set of *conditionally* independent *virtual branches*, and expressing the ordered-branch SNR variables as a linear function of virtual branch SNR variables. The key advantage of this formulation is that it allows greater flexibility in the selection process of the ordered instantaneous SNR values, and permits the combiner output SNR to be expressed in terms of the *conditionally independent* virtual branch SNR variables.

In this framework, the derivations of the mean of the combiner output SNR as well as the SEP of H-S/MRC, involving the evaluation of nested N-fold integrals, essentially reduce to the evaluation of a single integral. Note that the results for SC and MRC are special cases of our H-S/MRC results.

B. General Theory

Let γ_i denote the instantaneous SNR of the $i^{\underline{\text{th}}}$ diversity branch defined by

$$\gamma_i \triangleq \alpha_i^2 \frac{E_s}{N_{0i}}, \tag{1}$$

where E_s is the average symbol energy, α_i is the instantaneous gain and N_{0i} is the noise power spectral density of the $i^{\underline{\text{th}}}$ branch. We model the γ_i's as continuous random variables (r.v.'s) with probability density function (p.d.f.) $f_{\gamma_i}(x)$ and mean $\Gamma_i = \mathbb{E}\{\gamma_i\} = \mathbb{E}\{\alpha_i^2\} \frac{E_s}{N_{0i}} = \Omega_i \frac{E_s}{N_{0i}}$.

Let us first consider a general diversity-combining (GDC) system with the instantaneous output SNR of the form

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} a_i \gamma_{(i)}, \tag{2}$$

where $a_i \in \{0,1\}$, $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \ldots > \gamma_{(N)}$, and N is the number of available diversity branches. It will be apparent later that several diversity combining schemes, including H-S/MRC, turn out to be special cases of (2). Note that the possibility of at least two equal $\gamma_{(i)}$ is excluded, since $\gamma_{(i)} \neq \gamma_{(j)}$ *almost surely* for continuous r.v.'s γ_i.[2]

For a Rayleigh fading channel, the p.d.f. of the instantaneous branch SNR is given by

$$f_{\gamma_i}(x) = \begin{cases} \frac{1}{\Gamma_i} e^{-\frac{x}{\Gamma_i}}, & 0 < x < \infty \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Let $\mathcal{S}_N$ be the set of all permutations of integers $\{1, 2, \ldots, N\}$, and $\sigma \in \mathcal{S}_N$ denotes the particular function $\sigma : (1, 2, \ldots, N) \to (\sigma_1, \sigma_2, \ldots, \sigma_N)$ which permutes the integers $\{1, 2, \ldots, N\}$. Note that the cardinality of $\mathcal{S}_N$ is equal to $N!$. Denoting $\boldsymbol{\gamma}_{(N)} \triangleq (\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(N)})$, it can be shown that the joint p.d.f. of $\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(N)}$ for independent Rayleigh fading is

$$f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^N) = \begin{cases} \sum_{\sigma \in \mathcal{S}_N} \prod_{k=1}^{N} \frac{1}{\Gamma_{\sigma_k}} e^{-\frac{1}{\Gamma_{\sigma_k}} \gamma_{(k)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

It is important to note that the $\gamma_{(i)}$'s are *no* longer independent, even though the underlying γ_i's are independent.

Since the statistics of the ordered-branches are no longer independent, the analysis of the GDC system involves nested integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR of the ordered diversity branches into a new set of *virtual branch* instantaneous SNR's, V_n, using the following relation:

$$\gamma_{(k)} = \sum_{n=k}^{N} \left[\frac{1}{n} \sum_{m=1}^{n} \frac{1}{\Gamma_m} \right]^{-1} \frac{1}{n} V_n. \tag{5}$$

Denoting $\boldsymbol{V} \triangleq (V_1, V_2, \ldots, V_N)$, it can be verified that the r.v.'s $V_1, V_2, \ldots, V_N$ are *conditionally* independent with joint p.d.f. given by

$$f_{\boldsymbol{V}}(\{V_n\}_{n=1}^N) = \sum_{\sigma \in \mathcal{S}_N} \Pr\{\sigma\} \, f_{\boldsymbol{V}|\sigma}(\{V_n\}_{n=1}^N \mid \sigma) \tag{6}$$

where $f_{\boldsymbol{V}|\sigma}(\{V_n\}_{n=1}^N \mid \sigma)$ is the conditional p.d.f. of $\boldsymbol{V}$, conditioned upon the function σ, and

$$\Pr\{\sigma\} = \prod_{k=1}^{N} \frac{1}{\Gamma_{\sigma_k}} \left[\sum_{m=1}^{k} \frac{1}{\Gamma_{\sigma_m}} \right]^{-1}. \tag{7}$$

[2]In our context, the notion of "almost sure" or "almost everywhere" can be stated mathematically as: if $\mathcal{N} = \{\gamma_{(i)} = \gamma_{(j)}\}$, then $\Pr\{\mathcal{N}\} = 0$ [14], [15].

The conditional p.d.f., $f_{\mathbf{V}|\sigma}(\{V_n\}_{n=1}^N \mid \sigma)$, is given by

$$f_{\mathbf{V}|\sigma}(\{V_n\}_{n=1}^N \mid \sigma) = f_{V_1|\sigma}(V_1 \mid \sigma) f_{V_2|\sigma}(V_2 \mid \sigma) \dots f_{V_N|\sigma}(V_N \mid \sigma), \tag{8}$$

where

$$f_{V_n|\sigma}(V_n \mid \sigma) = \begin{cases} \frac{1}{\tilde{\Gamma}_n} e^{-\frac{V_n}{\tilde{\Gamma}_n}}, & 0 < V_n < \infty \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

with $\tilde{\Gamma}_n$ defined by

$$\tilde{\Gamma}_n \triangleq \left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_m}\right]\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}. \tag{10}$$

The characteristic function (c.f.) of V_n conditioned on σ is given by

$$\psi_{V_n|\sigma}(j\nu) \triangleq \mathbb{E}\left\{e^{+j\nu V_n} \mid \sigma\right\} = \frac{1}{1 - j\nu\tilde{\Gamma}_n}. \tag{11}$$

The instantaneous SNR of the combiner output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{\text{GDC}} = \sum_{n=1}^{N} b_n V_n, \tag{12}$$

where

$$b_n = \left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_m}\right]^{-1}\frac{1}{n}\sum_{m=1}^{n} a_m. \tag{13}$$

Using virtual branches, the derivation of the moments of the combiner output SNR essentially reduces to the calculation of the moments of the linear combination of virtual branch variables that are conditionally independent. The mean of the combiner output SNR can now be calculated using the chain rule of conditional expectations as:

$$\begin{aligned} \Gamma_{\text{GDC}} &= \mathbb{E}\left\{\sum_{n=1}^{N} b_n V_n\right\} = \sum_{n=1}^{N} b_n\, \mathbb{E}_\sigma\{\mathbb{E}\{V_n \mid \sigma\}\} \\ &= \sum_{n=1}^{N} b_n\, \mathbb{E}_\sigma\{\tilde{\Gamma}_n\} = \sum_{n=1}^{N} b_n \sum_{\sigma\in\mathcal{S}_N} \Pr\{\sigma\}\tilde{\Gamma}_n \\ &= \sum_{\sigma\in\mathcal{S}_N} \Pr\{\sigma\} \sum_{n=1}^{N} b_n \tilde{\Gamma}_n. \end{aligned} \tag{14}$$

Note that the conditional independence of the virtual branch variables plays a key role in simplifying the derivation of (14).

C. Symbol Error Probability for GDC over the Channel Ensemble

The SEP for GDC in a multipath-fading environment is obtained by averaging the conditional SEP over the channel ensemble. This can be accomplished by averaging $\Pr\{e \mid \gamma_{\text{GDC}}\}$ over the p.d.f. of γ_{GDC} as

$$\begin{aligned} P_{e,\text{GDC}} &= \mathbb{E}_{\gamma_{\text{GDC}}}\{\Pr\{e \mid \gamma_{\text{GDC}}\}\} \\ &= \int_0^\infty \Pr\{e \mid \gamma\}\, f_{\gamma_{\text{GDC}}}(\gamma) d\gamma, \end{aligned} \tag{15}$$

where $\Pr\{e \mid \gamma_{\text{GDC}}\}$ is the *conditional* SEP, conditioned on the r.v. γ_{GDC}, and $f_{\gamma_{\text{GDC}}}(\cdot)$ is the p.d.f. of the combiner output SNR. Alternatively, averaging over the channel ensemble can be accomplished, using the technique of [16], [17], [18], [19], by substituting the expression for γ_{GDC} directly in terms of the physical branch variables given in (2), as

$$\begin{aligned} P_{e,\text{GDC}} &= \mathbb{E}_{\{\gamma_{(i)}\}}\left\{\Pr\left\{e \mid \gamma_{\text{GDC}} = \textstyle\sum_{i=1}^N a_i\gamma_{(i)}\right\}\right\} \\ &= \int_0^\infty\int_0^{\gamma_{(1)}}\cdots\int_0^{\gamma_{(N-1)}} \Pr\left\{e \mid \textstyle\sum_{i=1}^N a_i\gamma_{(i)}\right\} \\ &\quad \times \quad f_{\gamma_{(N)}}(\{\gamma_{(i)}\}_{i=1}^N)\, d\gamma_{(N)}\dots d\gamma_{(2)}d\gamma_{(1)}. \end{aligned} \tag{16}$$

Note in (16) that, since the ordered physical branches are *no* longer independent, direct use of the methods given in [16], [17], [18], [19] requires an N-fold nested integration for the expectation operation in (16). This is alleviated using the virtual branch technique by substituting (12) into (15) and using the chain rule of conditional expectations as:

$$\begin{aligned} P_{e,\text{GDC}} &= \mathbb{E}_{\{V_n\}}\left\{\Pr\left\{e \mid \gamma_{\text{GDC}} = \textstyle\sum_{n=1}^N b_n V_n\right\}\right\} \\ &= \mathbb{E}_\sigma\left\{\mathbb{E}_{\{V_n|\sigma\}}\left\{\Pr\left\{e \mid \gamma_{\text{GDC}} = \textstyle\sum_{n=1}^N b_n V_n\right\} \,\middle|\, \sigma\right\}\right\} \\ &= \mathbb{E}_\sigma\{P_{\text{GDC}}(e \mid \sigma)\}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} P_{\text{GDC}}(e \mid \sigma) &\triangleq \mathbb{E}_{\{V_n|\sigma\}}\left\{\Pr\left\{e \mid \gamma_{\text{GDC}} = \textstyle\sum_{n=1}^N b_n V_n\right\} \,\middle|\, \sigma\right\} \\ &= \int_0^\infty\int_0^\infty\cdots\int_0^\infty \Pr\left\{e \mid \textstyle\sum_{n=1}^N b_n V_n\right\} \\ &\quad \times \quad \prod_{n=1}^{N} f_{V_n|\sigma}(V_n \mid \sigma)\, dV_n. \end{aligned} \tag{18}$$

For many important modulation techniques, it can be shown that $\Pr\left\{e \mid \sum_{n=1}^N b_n V_n\right\}$ factors into a product of N terms, where each term depends *only* on one of the V_n's. We will illustrate this by the following two important examples.

C.1 SEP for MPSK with GDC

For coherent detection of MPSK, an alternative representation for $\Pr\{e \mid \gamma_{\text{GDC}}\}$, involving a definite integral with *finite* limits, is given by [20], [21], [22]

$$\Pr\{e_{\text{MPSK}} \mid \gamma_{\text{GDC}}\} = \frac{1}{\pi}\int_0^\Theta e^{-\frac{c_{\text{MPSK}}}{2\sin^2\theta}\gamma_{\text{GDC}}} d\theta, \tag{19}$$

where $c_{\text{MPSK}} = 2\sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. Substituting (19) into (18), $P_{\text{GDC}}(e \mid \sigma)$ for MPSK becomes

$$\begin{aligned} P_{\text{GDC}}^{\text{MPSK}}(e \mid \sigma) &= \frac{1}{\pi}\int_0^{\Theta} \mathbb{E}_{\{V_n|\sigma\}}\left\{ e^{-\frac{c_{\text{MPSK}}}{2\sin^2\theta}\sum_{n=1}^{N} b_n V_n} \,\middle|\, \sigma \right\} d\theta \\ &= \frac{1}{\pi}\int_0^{\Theta} \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{-\frac{c_{\text{MPSK}}}{2\sin^2\theta}\sum_{n=1}^{N} b_n V_n} \\ &\times \prod_{n=1}^{N} f_{V_n|\sigma}(V_n \mid \sigma)\, dV_n\, d\theta . \end{aligned} \quad (20)$$

Exploiting the fact that the V_n's are independent, (20) becomes:

$$\begin{aligned} P_{\text{GDC}}^{\text{MPSK}}(e \mid \sigma) &= \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N} \mathbb{E}_{V_n|\sigma}\left\{ e^{-\frac{c_{\text{MPSK}} b_n}{2\sin^2\theta} V_n} \,\middle|\, \sigma \right\} d\theta \\ &= \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N} \psi_{V_n|\sigma}\left(-\frac{c_{\text{MPSK}} b_n}{2\sin^2\theta}\right) d\theta . \end{aligned} \quad (21)$$

The power of the virtual branch technique is apparent by observing that the expectation operation in (18) no longer requires an N-fold nested integration.

Using (11) in (21) gives

$$P_{\text{GDC}}^{\text{MPSK}}(e \mid \sigma) = \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\frac{b_n}{2}\tilde{\Gamma}_n + \sin^2\theta}\right] d\theta . \quad (22)$$

Substituting (22) into (17), we arrive at the SEP for coherent reception of MPSK using N-branch GDC as

$$P_{e,\text{GDC}}^{\text{MPSK}} = \sum_{\sigma \in \mathcal{S}_N} \Pr\{\sigma\}\, \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\frac{b_n}{2}\tilde{\Gamma}_n + \sin^2\theta}\right] d\theta . \quad (23)$$

Thus the derivation of the SEP for MPSK using N-branch GDC, involving the N-fold nested integrals in (16), essentially reduces to a single integral over θ with finite limits. The integrand is an N-fold product of a simple expression involving trigonometric functions. Note that the conditional independence of the virtual branch variables plays a key role in simplifying the derivation.

D. SEP for MQAM with GDC

For coherent detection of MQAM with $M = 2^k$ for even k, $\Pr\{e \mid \gamma_{\text{GDC}}\}$ is given by [19]

$$\begin{aligned} \Pr\{e_{\text{MQAM}} \mid \gamma_{\text{GDC}}\} &= q\,\frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{c_{\text{MQAM}}}{2\sin^2\theta}\gamma_{\text{GDC}}} d\theta \\ &- \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4} e^{-\frac{c_{\text{MQAM}}}{2\sin^2\theta}\gamma_{\text{GDC}}} d\theta , \end{aligned} \quad (24)$$

where $q = 4(1 - \frac{1}{\sqrt{M}})$, and $c_{\text{MQAM}} = \frac{3}{M-1}$. Using the virtual branch technique and a similar procedure as for MPSK, the SEP for MQAM becomes

$$\begin{aligned} P_{e,\text{GDC}}^{\text{MQAM}} = \sum_{\sigma \in \mathcal{S}_N} \Pr\{\sigma\} &\left\{ q\,\frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\text{MQAM}}\frac{b_n}{2}\tilde{\Gamma}_n + \sin^2\theta}\right] d\theta \right. \\ &\left. - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\text{MQAM}}\frac{b_n}{2}\tilde{\Gamma}_n + \sin^2\theta}\right] d\theta \right\} . \end{aligned} \quad (25)$$

Again, the derivation of the SEP for coherent reception of MQAM using GDC in Rayleigh fading reduces to two terms, each consisting of a single integral over θ involving trigonometric functions with finite limits.

III. Application of General Theory

A. Hybrid S/MRC Analysis

In this section, the general theory derived in Section II-C is used to evaluate the performance of H-S/MRC. The instantaneous output SNR of H-S/MRC is

$$\gamma_{\text{S/MRC}} = \sum_{i=1}^{L} \gamma_{(i)} , \quad (26)$$

where $1 \le L \le N$. Note that $\gamma_{\text{S/MRC}} = \gamma_{\text{GDC}}$ with

$$a_i = \begin{cases} 1, & i = 1, \dots, L \\ 0, & \text{otherwise}. \end{cases} \quad (27)$$

In this case,

$$b_n\tilde{\Gamma}_n = \begin{cases} \left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}, & n \le L \\ \left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}\frac{L}{n}, & \text{otherwise}. \end{cases} \quad (28)$$

Substituting (28) into (14), the mean of the combiner output SNR for H-S/MRC can be obtained as

$$\begin{aligned} \Gamma_{\text{S/MRC}} = \sum_{\sigma \in \mathcal{S}_N} \Pr\{\sigma\} &\left\{ \sum_{n=1}^{L}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1} \right. \\ &\left. + \; L\sum_{n=L+1}^{N}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1} \right\} . \end{aligned} \quad (29)$$

Substituting (28) into (23) and (25), the SEP for H-S/MRC can be obtained as

$$P_{e,\text{S/MRC}} = \sum_{\sigma \in \mathcal{S}_N} \Pr\{\sigma\}\, P_{\text{S/MRC}}(e \mid \sigma) , \quad (30)$$

where the expressions for $P_{\text{S/MRC}}(e \mid \sigma)$ for MPSK and MQAM are given respectively by

$$\begin{aligned} &P_{\text{S/MRC}}^{\text{MPSK}}(e \mid \sigma) \\ &= \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{L}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1} + \sin^2\theta}\right] \\ &\times \prod_{n=L+1}^{N}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\frac{L}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1} + \sin^2\theta}\right] d\theta , \end{aligned} \quad (31)$$

and

$$
\begin{aligned}
&P_{\mathrm{S/MRC}}^{\mathrm{MQAM}}(e \mid \sigma) \\
&= q\,\frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{L}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right] \\
&\times\prod_{n=L+1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{L}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta \\
&-\frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\prod_{n=1}^{L}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right] \\
&\times\prod_{n=L+1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{L}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta\,. \qquad (32)
\end{aligned}
$$

B. Limiting Case 1: SC System

SC is the simplest form of diversity combining whereby the received signal from *one* of N diversity branches is selected [2]. The output SNR of SC is

$$\gamma_{\mathrm{SC}} = \max_i\{\gamma_i\}\,. \qquad (33)$$

Note that $\gamma_{\mathrm{SC}} = \gamma_{\mathrm{GDC}}$ with $a_1 = 1$ and $a_i = 0$ for $i = 2, \ldots, N$. In this case,

$$b_n\tilde{\Gamma}_n = \left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}. \qquad (34)$$

Substituting (34) into (14), the mean of the combiner output SNR for SC becomes

$$\Gamma_{\mathrm{SC}} = \sum_{\sigma\in\mathcal{S}_N}\Pr\{\sigma\}\left\{\sum_{n=1}^{N}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}\right\}. \qquad (35)$$

Substituting (34) into (23) and (25), the SEP for SC can be obtained as

$$P_{e,\mathrm{SC}} = \sum_{\sigma\in\mathcal{S}_N}\Pr\{\sigma\}\,P_{\mathrm{SC}}(e \mid \sigma)\,, \qquad (36)$$

where where the expressions for $P_{\mathrm{SC}}(e \mid \sigma)$ for MPSK and MQAM are given respectively by

$$
\begin{aligned}
&P_{\mathrm{SC}}^{\mathrm{MPSK}}(e \mid \sigma) \\
&= \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MPSK}}\frac{1}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta\,, \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
&P_{\mathrm{SC}}^{\mathrm{MQAM}}(e \mid \sigma) \\
&= q\,\frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta \\
&-\frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta\,. \qquad (38)
\end{aligned}
$$

Alternatively, (35), (37), and (38) can also be obtained from the H-S/MRC results (29), (31) and (32) by setting $L = 1$. This should be expected since SC is a limiting case of H-S/MRC with $L = 1$.

C. Limiting Case 2: MRC System

In MRC, the received signals from *all* diversity branches are weighted and combined to maximize the SNR at the combiner output. The output SNR of MRC is given by

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{N}\gamma_{(i)}\,. \qquad (39)$$

Note that $\gamma_{\mathrm{MRC}} = \gamma_{\mathrm{GDC}}$ with $a_i = 1 \quad \forall i$. In this case,

$$b_n\tilde{\Gamma}_n = \left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}. \qquad (40)$$

Substituting (40) into (14), the mean of the combiner output SNR for MRC becomes

$$\Gamma_{\mathrm{MRC}} = \sum_{\sigma\in\mathcal{S}_N}\Pr\{\sigma\}\left\{\sum_{n=1}^{N}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}\right\}. \qquad (41)$$

Substituting (40) into (23) and (25), the SEP for MRC can be obtained as

$$P_{e,\mathrm{MRC}} = \sum_{\sigma\in\mathcal{S}_N}\Pr\{\sigma\}\,P_{\mathrm{MRC}}(e \mid \sigma)\,, \qquad (42)$$

where the expressions for $P_{\mathrm{MRC}}(e \mid \sigma)$ for MPSK and MQAM are given respectively by

$$
\begin{aligned}
&P_{\mathrm{MRC}}^{\mathrm{MPSK}}(e \mid \sigma) \\
&= \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MPSK}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta\,, \qquad (43)
\end{aligned}
$$

and

$$
\begin{aligned}
&P_{\mathrm{MRC}}^{\mathrm{MQAM}}(e \mid \sigma) \\
&= q\,\frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta \\
&-\frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\frac{1}{2}\left[\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\Gamma_{\sigma_m}}\right]^{-1}+\sin^2\theta}\right]d\theta. \quad (44)
\end{aligned}
$$

Note again that the results for MRC given in (41), (43), and (44) may also be obtained from the H-S/MRC results (29), (31) and (32) by setting $L = N$ since MRC is a limiting case of H-S/MRC with $L = N$.

IV. Conclusions

We derived the mean of the combiner output signal-to-noise ratio (SNR) as well as the symbol error probability (SEP) of a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system in a multipath fading environment. In particular, we considered independent Rayleigh fading on each diversity branch with not necessarily equal SNR's, averaged over the fading. We analyzed this system using a "virtual branch" technique which resulted in a simple derivation and formulas for *any* L and N. The key idea was to transform the dependent ordered-branch variables into a new set of conditionally independent *virtual branches*, and express the combiner output SNR as a linear combination of the *conditionally independent* virtual branch SNR variables. In this framework, the mean of the combiner output SNR as well as the SEP of H-S/MRC were derived succinctly. The results for SC and MRC were obtained as special cases of our H-S/MRC results. Our results allow easy analysis of the improved performance for $L < N$ of H-S/MRC over L branch MRC.

Acknowledgments

The authors wish to thank L. A. Shepp, C. L. Mallows, G. Chrisikos, J. G. Proakis, N. R. Sollenberger, G. J. Foschini, and P. F. Dahm for helpful discussions.

References

[1] Jack H. Winters, Jack Salz, and Richard Gitlin, "The impact of antenna diversity on the capacity of wireless communication system," *IEEE Trans. Commun.*, vol. 42, no. 2/3/4, pp. 1740–1751, Feb./Mar./Apr. 1994.

[2] William C. Jakes, Ed., *Microwave Mobile Communications*, IEEE Press, Piscataway, New Jersey, 08855-1331, IEEE press classic reissue edition, 1995.

[3] Jack H. Winters, "Smart antennas for wireless systems," *IEEE Person. Commun. Mag.*, vol. 42, no. 2/3/4, pp. 6–10, Feb. 1998.

[4] Raymond L. Pickholtz, Donald L. Schilling, and Laurence B. Milstein, "Theory of spread-spectrum communications – A tutorial," *IEEE Trans. Commun.*, vol. COM-30, no. 5, pp. 855–884, May 1982.

[5] Moe Z. Win and Robert A. Scholtz, "On the energy capture of ultra-wide bandwidth signals in dense multipath environments," *IEEE Commun. Lett.*, vol. 2, no. 9, pp. 245–247, Sept. 1998.

[6] Gwo-Tsuey Chyi, John G. Proakis, and Katherine M. Keller, "On the symbol error probability of maximum-selection diversity reception schemes over a Rayleigh fading channel," *IEEE Trans. Commun.*, vol. 37, no. 1, pp. 79–83, Jan. 1989.

[7] Elisabeth A. Neasmith and Norman C. Beaulieu, "New results on selection diversity," *IEEE Trans. Commun.*, vol. 46, no. 5, pp. 695–704, May 1998.

[8] John G. Proakis, *Digital Communications*, McGraw-Hill, Inc., New York, NY, 10020, third edition, 1995.

[9] Norman C. Beaulieu and Adnan A. Abu-Dayya, "Analysis of equal gain diversity on Nakagami fading channels," *IEEE Trans. Commun.*, vol. 39, no. 2, pp. 225–234, Feb. 1991.

[10] H. Erben, S. Zeisberg, and H. Nuszkowski, "BER performance of a hybrid SC/MRC 2DPSK RAKE receiver in realistic mobile channels," in *Proc. 44th Annual Int. Veh. Technol. Conf.*, June 1994, vol. 2, pp. 738–741, Stockholm, Sweden.

[11] Thomas Eng, Ning Kong, and Laurence B. Milstein, "Comparison of diversity combining techniques for Rayleigh-fading channels," *IEEE Trans. Commun.*, vol. 44, no. 9, pp. 1117–1129, Sept. 1996.

[12] Ning Kong and Laurence B. Milstein, "Combined average SNR of a generalized diversity selection combining scheme," in *Proc. IEEE Int. Conf. on Comm.*, June 1998, vol. 3, pp. 1556–1560, Atlanta, GA.

[13] Moe Z. Win and Jack H. Winters, "Analysis of hybrid selection/maximal-ratio combining in Rayleigh fading," in *Proc. IEEE Int. Conf. on Comm.*, June 1999, Vancouver, Canada, to appear.

[14] Al'bert Nikolaevich Shiryaev, *Probability*, Springer-Verlag, New York, second edition, 1995.

[15] Richard Durrett, *Probability: Theory and Examples*, Wadsworth and Brooks/Cole Publishing Company, Pacific Grove, California, first edition, 1991.

[16] J. W. Craig, "A new, simple and exact result for calculating the probability of error for two-dimensional signal constellations," in *Proc. Military Comm. Conf.*, 1991, pp. 25.5.1–25.5.5, Boston, MA.

[17] C. Tellambura, A. Joseph Mueller, and Vijay Bhargava, "Analysis of M-ary phase-shift keying with diversity reception for land-mobile satellite channels," *IEEE Trans. on Vehicul. Technol.*, vol. 46, no. 4, pp. 910–922, Nov. 1997.

[18] Marvin K. Simon and Dariush Divsalar, "Some new twists to problems involving the Gaussian probability integral," *IEEE Trans. Commun.*, vol. 46, no. 2, pp. 200–210, Feb. 1998.

[19] Mohamed-Slim Alouini and Andrea Goldsmith, "A unified approach for calculating error rates of linearly modulated signals over generalized fading channels," in *Proc. IEEE Int. Conf. on Comm.*, June 1998, vol. 1, pp. 459–463, Atlanta, GA.

[20] R. F. Pawula, S. O. Rice, and J. H. Roberts, "Distribution of the phase angle between two vectors perturbed by Gaussian noise," *IEEE Trans. Commun.*, vol. COM-30, no. 8, pp. 1828–1841, Aug. 1982.

[21] R. R. Pawula, "A new formula for MDPSK symbol error probability," *IEEE Commun. Lett.*, vol. 2, no. 10, pp. 271–272, Oct. 1998.

[22] Marvin K. Simon, Sami M. Hinedi, and William C. Lindsey, *Digital Communication Techniques: Signal Design and Detection*, Prentice Hall, Englewood Cliffs, New Jersey 07632, first edition, 1995.