On Maximal Ratio Combining in Correlated Nakagami Channels with Unequal Fading Parameters and SNR's Among Branches: An Analytical Framework

Moe Z. Win[†], *Senior Member, IEEE,* Jack H. Winters, *Fellow, IEEE*

Wireless Systems Research Department, AT&T Labs - Research
NSL 4-147, 100 Schulz Drive, Red Bank, NJ 07701-7033 USA
Tel: (732)-345-3144 Fax: (732)-345-3038
e-mail: {win, jhw}@research.att.com

Abstract— **In this paper, we develop an analytical framework to study the performance of wireless systems using maximal ratio combining (MRC) with an arbitrary number of diversity branches in correlated multipath fading. We consider the coherent detection of digital signals received over correlated Nakagami fading channels, where the instantaneous signal-to-noise ratios (SNR's) of the diversity branches are *not* necessarily independent *or* identically distributed. Specifically: 1) these SNR's can be arbitrarily correlated; 2) the SNR distributions can be from different Nakagami families, i.e., fading parameters (m's) are not necessarily equal; and 3) the average SNR's (averaged over the fading) of the branches are not necessarily equal.**

We derive closed-form expressions for three performance measures of a MRC diversity system: 1) probability density function (p.d.f.) of the combiner output SNR; 2) symbol error probability (SEP) for coherent detection; and 3) outage probability. We obtain a *canonical structure* for these performance measures as a weighted sum of the corresponding expressions for a non-diversity (single-branch) system with appropriately-defined parameters. This result is fundamental: the canonical structure depends *only* on the properties of the channel and diversity combiner, and *not* on the specific modulation technique. Calculations of SEP for specific modulation techniques are illustrated through examples.

I. Introduction

MAXIMAL ratio combining (MRC) has been known to improve the reliability of transmission systems for more than four decades [1], [2], [3]. With maximal ratio combining, the received signals from multiple diversity branches are cophased, weighted, and combined to maximize the output signal-to-noise ratio (SNR). These diversity branches can be colocated antenna arrays, widely spatially-separated antennas of a macrodiversity system, frequency bins of a channelized receiver, or fingers of a Rake receiver in a wireless communication system, where MRC mitigates the effect of multipath fading. Early work on the evaluation of symbol error probability (SEP) of MRC has mainly concentrated on Rayleigh and Rician channels [4], [5], [3].

Recently, Nakagami fading channels have received considerable attention in the study the various aspects of wireless systems [6], [7], [8], [9], [10], [11], [12], [13]. The Nakagami distribution, also known as the "m-distribution," provides greater flexibility in matching experimental data. Experimental results have shown that the Nakagami distribution fits experimental data collected in a variety of fading environments better than Rayleigh, Rician, or log-normal distributions [14], [15], [16]. A comprehensive description of the Nakagami distribution is given in [17], and the derivation and physical insights of the Nakagami-fading model can be found in [16]. The Nakagami family of distributions span from the one-sided Gaussian distribution (m=1/2) to the non-fading channel case ($m = \infty$), and contain Rayleigh fading (m=1) as a special case; along with the cases of fades that are more severe than Rayleigh ($1/2 \leq m < 1$) and fades that are less severe than Rayleigh ($1 < m$). They can also be used as an approximation to log-normal and Rician distributions for a certain range of average SNR's [8]. Furthermore, the Nakagami distribution offers analytical convenience since it is a "central" distribution.

Closed-form expressions for the error probability of MRC in Nakagami fading using independent branches, with the same fading parameter "m," but different average SNR, on each branch, were obtained in [11] by approximating the sum of the squares of Nakagami random variables (r.v.'s) by another Nakagami r.v. with appropriate parameters. This was extended to the case of independent fading with unequal ratios of m to average SNR for *all* diversity branches in [13], where the result requires the evaluation of a single integral with *infinite* limits. Performance of MRC with independent diversity branches, for integer m and unequal ratios of m to average SNR for *all* diversity branches, was given in [6]. The analysis of MRC in corre-

Moe Z. Win and Jack H. Winters are with the Wireless Systems Research Department, Newman Springs Laboratory, AT&T Labs - Research, 100 Schulz Drive, Red Bank, NJ 07701-7033 USA (e-mail: win@research.att.com, jhw@research.att.com)
[†]Corresponding author.

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lated fading has been limited in previous papers to dual-branch diversity [6], with the exception of [12]. The study in [12], though, assumed two specific correlation models, namely the equal-correlation and exponential-correlation models; with equal m as well as average SNR's among diversity branches. We point out in passing that these studies were done for a limited number of specific modulation techniques.

However, in some cases the average received signal power is not equal for all the diversity branches and the fading statistics can also be different for each diversity branch. Such cases include MRC with: 1) angle diversity using multiple beams, where the average signal strength and fading statistics can be different in each beam; 2) polarization diversity using horizontal and vertical polarization with high base station antennas, where for a vertically-polarized transmitter the average received signal strength is typically 6 to 10 dB lower for the horizontally-polarized antenna; 3) macrodiversity, where the shadow fading is different at each antenna and/or the local ports are spaced far enough such that different local scattering conditions lead to different fading statistics; and 4) Rake receivers, where the distribution of signal power with delay is not uniform, and the first arriving multipath component is more likely to contain a specular component (corresponding to larger m) and later components are more diffuse ($m \approx 1$). In these cases, closed-form expressions for the performance of MRC are not previously available in the literature.

In this paper, we develop an analytical framework to study the performance of wireless systems using MRC with an arbitrary number of branches in correlated-fading environments. We consider coherent detection of digital signals over correlated Nakagami fading channels, where the instantaneous SNR values of the diversity branches are *not* necessarily independent *or* identically distributed. The proposed problem is made analytically tractable by transforming the physical diversity branches into the "virtual branch" domain. Note that we used the virtual branch technique in [18] to determine the mean and variance of the combiner output SNR for hybrid selection/maximal-ratio combining. By averaging the conditional SEP over the individual probability density function (p.d.f.) of the virtual branch SNR's, we derived the SEP in [19] for coherent reception of digital signals using MRC. The striking resemblance between the SEP expressions obtained in [19] for the two specific examples, namely phase-shift keying (MPSK) and M-ary quadrature amplitude modulation (MQAM), is the compelling impetus for our study in this paper.

We derive the p.d.f. of the combiner output SNR. The *canonical structure* of this p.d.f. emerges from our derivation as a weighted sum of elementary p.d.f.'s, which are the p.d.f.'s of the single-branch SNR with appropriate fading parameters and average SNR. This allows the derivation of a closed-form expression of the SEP for *arbitrary* modulation techniques, which is simply the weighted sum of the elementary SEP's, namely the SEP for the single-branch reception. Finally, we derive the canonical structure for the outage probability, where, similar to the SEP case, the outage probability is a weighted sum of the outage probabilities for the single-branch reception in Nakagami channels with appropriately-defined parameters.

II. Diversity Combining Analysis

A. Preliminaries

Consider N-branch diversity reception in a correlated-fading environment. The equivalent lowpass (ELP) version of the $i^{\underline{\text{th}}}$ branch output is given by

$$r_i(t) = \alpha_i s_i(t) + n_i(t)\,, \qquad i = 1, \ldots, N, \tag{1}$$

where $n_i(t)$ is an additive white Gaussian[1] noise (AWGN) process, assumed to be independent of the received signal, with two-sided power spectral density N_{0i}, $s_i(t)$ is the information-bearing signal with the average symbol energy E_s, and α_i is the $i^{\underline{\text{th}}}$ diversity branch gain. We model the α_i's as correlated Nakagami r.v.'s with a *marginal* p.d.f. given by

$$f_{\alpha_i}(r) = \frac{2}{\Gamma(m_i)} \left(\frac{m_i}{\Omega_i}\right)^{m_i} r^{2m_i - 1} e^{-m_i r^2/\Omega_i}, \tag{2}$$

where the fading parameter m_i denotes the Nakagami family, and $\Omega_i = \mathbb{E}\left\{\alpha_i^2\right\}$. Note that the α_i's can be from different Nakagami families, where m_i and Γ_i are not necessarily equal among the branches. We will refer to $(m_i, \Omega_i \frac{E_s}{N_{0i}})$ as Nakagami parameter pairs. As in [9], we assume that the m_i's are integers, noting that the measurement accuracy of the channel is typically only of integer order.

The instantaneous output SNR with MRC is given by [3]

$$\gamma_{\text{MRC}} = \sum_{i=1}^{N} \gamma_i\,, \tag{3}$$

where γ_i denotes the instantaneous SNR of the $i^{\underline{\text{th}}}$ diversity branch defined by $\gamma_i \triangleq \alpha_i^2 \frac{E_s}{N_{0i}}$. For a correlated Nakagami fading channel, the marginal p.d.f. of γ_i is given by

$$g_{\gamma_i}(x; m_i/\Gamma_i, m_i) = \frac{1}{\Gamma(m_i)} \left(\frac{m_i}{\Gamma_i}\right)^{m_i} x^{m_i - 1} e^{-m_i x/\Gamma_i}, \tag{4}$$

where the SNR averaged over the fading in the $i^{\underline{\text{th}}}$ branch $\Gamma_i = \mathbb{E}\{\gamma_i\} = \mathbb{E}\left\{\alpha_i^2\right\} \frac{E_s}{N_{0i}} = \Omega_i \frac{E_s}{N_{0i}}$. The family of distributions with p.d.f.'s of the form given in (4) is referred to as the gamma family of distributions.[2]

B. Virtual Branch Technique: The Key Idea

Conventional analysis of MRC in correlated Nakagami fading is, in general, cumbersome and complicated since: 1) the γ_i's are correlated; 2) the γ_i's can be from different

[1] The term "Gaussian" is used to denote the "ELP complex circular Gaussian."

[2] In general, the gamma density is denoted by $g_{\gamma_i}(x; \alpha, p)$, where $\mathbb{E}\{\gamma_i\} = \frac{p}{\alpha}$, and $Var\{\gamma_i\} = \frac{p}{\alpha^2}$. The special case of a gamma distribution with $\alpha = \frac{1}{2}$ and $p = \frac{k}{2}$ is a chi-squared distribution with k degrees of freedom [20].

Nakagami families where the m_i's are not necessarily equal; and 3) the Γ_i's are not necessarily equal.[3] The difficulty described above is alleviated in the following by transforming the dependent physical branch variables into a new set of independent *virtual branches* and expressing the combiner output SNR as a linear function of the independent virtual branch SNR's.

Let X_i be the $2m_i \times 1$ vector defined by

$$X_i \triangleq \begin{bmatrix} X_{i,1} \\ X_{i,2} \\ \vdots \\ X_{i,2m_i} \end{bmatrix}, \qquad i = 1, 2, \ldots, N, \tag{5}$$

where the elements of X_i, $X_{i,k}$'s, are independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance $\mathbb{E}\left\{X_{i,k}^2\right\} = \frac{\Gamma_i}{2m_i}$. It can be shown that each γ_i is infinitely divisible [20], [21], [22]. The infinite divisibility has implications on the *statistical* representation of γ_i as $\gamma_i \stackrel{\text{L}}{=} X_i^t X_i$, where the notation "$\stackrel{\text{L}}{=}$" denotes "equal in their respective distributions" (or "equal in their respective Laws").[4] Therefore

$$\gamma_{\text{MRC}} \stackrel{\text{L}}{=} \sum_{i=1}^{N} X_i^t X_i = X^t X, \tag{6}$$

where X is the $D_T \times 1$ vector defined by

$$X \triangleq \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \tag{7}$$

and $D_T = \sum_{i=1}^{N} 2m_i$ denotes twice the sum of the Nakagami parameters. In general, the p.d.f. of γ_{MRC} is *not* equal to $g_{\gamma_{\text{MRC}}}(x; \beta, \frac{D_T}{2})$. However, for the special case when the $\{X_i\}$'s are independent *and* the $\frac{m_i}{\Gamma_i}$'s are all equal ($\frac{m_i}{\Gamma_i} = \beta$ for some constant β), then the p.d.f. of γ_{MRC} is $g_{\gamma_{\text{MRC}}}(x; \beta, \frac{D_T}{2})$ [20], [23].

The statistical dependence among the N correlated branches can be related to the statistical dependence among the elements of X, by carefully constructing X. When there is only second-order dependence, it suffices to construct the covariance matrix of X given by $K_X = \mathbb{E}\{XX^t\}$. Without loss of generality, one can assume that the γ_i's are indexed in increasing order of their Nakagami parameters, i.e., $m_1 \leq m_2 \leq \ldots \leq m_N$. We construct the correlation among the elements X such that

$$\mathbb{E}\{X_{i,k}X_{j,l}\} = \begin{cases} \frac{\Gamma_i}{2m_i} & \text{if } i = j \text{ and } k = l \\ \rho_{i,j}\sqrt{\frac{\Gamma_i}{2m_i}\frac{\Gamma_j}{2m_j}}, & \text{if } i \neq j \text{ but} \\ & k = l = 1, 2, \ldots, 2\min\{m_i, m_j\} \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

This construction implies that the $k^{\underline{\text{th}}}$ entries of X_i and X_j, with $i \neq j$, are correlated for $k = 1, 2, \ldots, 2\min\{m_i, m_j\}$. However, all the entries of X_i are mutually independent, and all other entries are independent.

The relationship between the covariance of γ_i and γ_j and the covariance of the elements of X is given by:

$$\rho_{\gamma_i\gamma_j} \triangleq \frac{\mathbb{E}\{(\gamma_i - \mathbb{E}\{\gamma_i\})(\gamma_j - \mathbb{E}\{\gamma_j\})\}}{\sqrt{Var\{\gamma_i\}Var\{\gamma_j\}}} = \sqrt{\frac{\min\{m_i, m_j\}}{\max\{m_i, m_j\}}}\,\rho_{i,j}^2. \tag{9}$$

The lower and upper bounds for the correlation between the two Nakagami branches are given by $0 \leq \rho_{\gamma_i\gamma_j} \leq \sqrt{\frac{\min\{m_i, m_j\}}{\max\{m_i, m_j\}}}$.[5]

Let $\{\lambda_l\}$ be the set of L *distinct* eigenvalues of K_X where each λ_l has algebraic multiplicity μ_l such that $\sum_{l=1}^{L}\mu_l = D_T = \sum_{i=1}^{N} 2m_i$. The corresponding *orthonormal* eigenvectors are denoted by $\{\phi_{l,k}\}$. Then the Karhunen-Loève (KL) expansion of the vector X is [24]

$$X = \sum_{l=1}^{L}\sqrt{\lambda_l}\sum_{k=1}^{\mu_l} W_{l,k}\phi_{l,k} \tag{10}$$

where the $\{W_{l,k}\}$'s are *independent* zero-mean unity-variance Gaussian r.v.'s. A similar technique employing a frequency-domain KL expansion was used in [25] to study diversity combining in a frequency-selective Rayleigh-fading channel. Another technique similar to KL expansion was also used in [26] to study the reception of noncoherent orthogonal signals in Rician and Rayleigh fading channels. Using (10), the combiner output SNR can be described in a statistically equivalent representation as

$$\gamma_{\text{MRC}} \stackrel{\text{L}}{=} \sum_{l=1}^{L}\lambda_l\sum_{k=1}^{\mu_l} W_{l,k}^2 = \sum_{l=1}^{L}\lambda_l V_l, \tag{11}$$

where the virtual branch variables V_l's are defined by

$$V_l \triangleq \sum_{k=1}^{\mu_l} W_{l,k}^2. \tag{12}$$

Exploiting the fact that the $\{W_{l,k}\}$'s in the KL expansion are independent zero-mean unity-variance Gaussian r.v.'s,

[3]Note that our model includes the case where only a *proper* subset of the branches have the same m_i's and/or Γ_i's. This is a subtle but important difference with previous studies where the analyses given in [6] and [12] required that the $\frac{m_i}{\gamma_i}$'s are either *all* different or *all* equal.

[4]We stress that, in general $\gamma_i \neq X_i^t X_i$ and the notation "$\stackrel{\text{L}}{=}$" is used to merely indicate that only the respective distributions (or Laws) are equal [20], [21], [22]. One can view $X_i^t X_i$ as *a statistical* representation of γ_i, and both forms can be used interchangeably in performing statistical analyses.

[5]The fact that two Nakagami branches with *different* fading parameters m_i and m_j can not be completely correlated (i.e., $\rho_{\gamma_i\gamma_j} < 1$) is not a drawback in our statistical representation, and it is just a manifestation of the basic fact that two r.v.'s with different distributions can not be completely correlated.

it can be shown that the V_l's are *independent* chi-squared r.v.'s with μ_l degrees of freedom. Therefore the characteristic function (c.f.) of the V_l is given by

$$\psi_{V_l}(j\nu) \triangleq \mathbb{E}\left\{e^{+j\nu V_l}\right\} = \left[\frac{1/2}{1/2 - j\nu}\right]^{\mu_l/2}. \quad (13)$$

III. Canonical Forms

The SEP for *specific* modulation techniques, namely MPSK and MQAM, in correlated Nakagami fading was derived in [19] by averaging the conditional SEP over the individual p.d.f. of the virtual branch SNR's (rather than averaging over the p.d.f. of the combined output SNR). In particular, the results derived in [19] showed that the SEP's for MPSK and MQAM reception using N-branch MRC with unequal branch SNR in correlated Nakagami fading are simply the weighted sum of the single-branch SEP's. The striking resemblance between the canonical structures of SEP for MPSK and MQAM suggests the more profound result that a canonical structure exists, and that it is the same structure for *all* modulation techniques.

A. Canonical Form for p.d.f. of the Combiner Output SNR

The hypothesis that the canonical structure is independent of modulation techniques implies that *it is the property of both the channel and the diversity combiner and not the property of specific modulation techniques.* The most fundamental statistical quantity describing this property is the p.d.f. of the combiner output SNR. Therefore, we derive the canonical structure for the p.d.f. of the combiner output SNR in the following by inverting the c.f. The c.f. of the combiner output SNR is

$$\psi_{\gamma_{\mathrm{MRC}}}(j\nu) = \mathbb{E}\left\{e^{+j\nu\gamma_{\mathrm{MRC}}}\right\}. \quad (14)$$

Since the physical branches are *correlated*, the direct use of the expression for γ_{MRC} in terms of physical branch variables given by (3) requires N-fold integration for the evaluation of the c.f. of γ_{MRC} in (14). This is alleviated by expressing γ_{MRC} in terms of virtual branch variables using (11) as:

$$\begin{aligned}\psi_{\gamma_{\mathrm{MRC}}}(j\nu) &= \mathbb{E}\left\{e^{+j\nu\sum_{l=1}^{L}\lambda_l V_l}\right\} \\ &= \prod_{l=1}^{L} \underbrace{\mathbb{E}\left\{e^{+j\nu\lambda_l V_l}\right\}}_{\triangleq\, \psi_{V_l}(j\nu\lambda_l)}. \quad (15)\end{aligned}$$

The power of the virtual path technique is apparent by observing that the expectation operation in the above equation only requires a single integral, instead of N-fold integration. Substituting (13) into (15) gives

$$\psi_{\gamma_{\mathrm{MRC}}}(j\nu) = \prod_{l=1}^{L}\left[\frac{1/2}{1/2 - j\nu\lambda_l}\right]^{\mu_l/2}. \quad (16)$$

It can be shown that the carefully constructed statistical equivalence of γ_{MRC} in terms of X with K_X in Section II-B guarantees that μ_l is even. Expanding (16) using partial fraction expansion, the p.d.f. of the combiner output SNR can be derived as

$$\begin{aligned}f_{\gamma_{\mathrm{MRC}}}(\gamma) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_{\gamma_{\mathrm{MRC}}}(j\nu)e^{-j\nu\gamma}\,d\nu \\ &= \sum_{l=1}^{L}\sum_{k=1}^{\mu_l/2} A_{l,k}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty}\underbrace{\left[\frac{\frac{1}{2\lambda_l}}{\frac{1}{2\lambda_l} - j\nu}\right]^k}_{\triangleq\psi(j\nu)} e^{-j\nu\gamma}\,d\nu, \quad (17)\end{aligned}$$

where the $A_{l,k}$'s are partial fraction expansion coefficients. It can be shown that

$$f_{\gamma_{\mathrm{MRC}}}(\gamma) = \sum_{l=1}^{L}\sum_{k=1}^{\mu_l/2} A_{l,k}\; g_\gamma(\gamma; \frac{1}{2\lambda_l}, k), \quad (18)$$

and $g_\gamma(\gamma; \frac{1}{2\lambda_l}, k)$ is the p.d.f. of the single-branch SNR in Nakagami fading with parameter pair $(k, 2k\lambda_l)$.

The result given in (18) is the p.d.f. of the combiner output SNR for N-branch MRC in correlated Nakagami channels. Specifically, the N diversity branches are correlated, where m_i and Γ_i are not necessarily equal among the branches. The canonical form for the p.d.f. of the combiner output SNR is evident from (18): it is simply the weighted sum of the "elementary p.d.f.'s," where the elementary p.d.f.'s are the p.d.f.'s of the single-branch SNR in Nakagami fading with parameter pair $(k, 2k\lambda_l)$. Note also that the elementary p.d.f. is equivalent to the p.d.f. of the combiner output SNR in Rayleigh-fading channels using MRC with k *identical* branches having *equal* SNR's of $2\lambda_l$.

B. Canonical Form for SEP with Arbitrary Modulation Techniques

In Section III-A, we derived the p.d.f. of the combiner output SNR. Using the canonical structure of the p.d.f. of the combiner output SNR given in (18), we show in the following that the canonical structure for SEP exists and is *independent* of the specific modulation techniques used.

The SEP for MRC in correlated Nakagami fading can be obtained by averaging the $\Pr\{e|\gamma_{\mathrm{MRC}}\}$ (for any specific modulation technique) over the p.d.f. of the γ_{MRC} as

$$P_e = \mathbb{E}_{\gamma_{\mathrm{MRC}}}\left\{\Pr\{e|\gamma_{\mathrm{MRC}}\}\right\}, \quad (19)$$

where $\Pr\{e|\gamma_{\mathrm{MRC}}\}$ is the SEP conditioned on the random variable γ_{MRC}. Substituting (18) into (19),

$$\begin{aligned}P_e &= \int_0^{+\infty}\Pr\{e|\gamma\}\, f_{\gamma_{\mathrm{MRC}}}(\gamma)\,d\gamma \\ &= \sum_{l=1}^{L}\sum_{k=1}^{\mu_l/2} A_{l,k}\underbrace{\int_0^{+\infty}\Pr\{e|\gamma\}\, g_\gamma(\gamma; \frac{1}{2\lambda_l}, k)\,d\gamma}_{\triangleq\, P_e(k, 2k\lambda_l)}. \quad (20)\end{aligned}$$

Recognizing that the integral in (20), denoted by $P_e(k, 2k\lambda_l)$, is the average of the conditional SEP's (for any specific modulation technique) over $g_\gamma(\gamma; \frac{1}{2\lambda_l}, k)$, we obtain

$$P_e = \sum_{l=1}^{L} \sum_{k=1}^{\mu_l/2} A_{l,k} \; P_e(k, 2k\lambda_l)\,. \qquad (21)$$

In other words, the SEP (for any specific modulation technique) for MRC with N branches, where m_i and Γ_i are not necessarily equal among the branches, in correlated Nakagami fading is simply the weighted sum of the "elementary SEP's" for that modulation technique. The elementary SEP's are the SEP for single-branch reception in Nakagami fading with parameter pair $(k, 2k\lambda_l)$, or equivalently the SEP in Rayleigh-fading channels using MRC with k *identical* branches having *equal* SNR's of $2\lambda_l$.

B.1 SEP Calculation Examples

The canonical form for SEP derived in Section III-B can be used to evaluate the SEP for a variety of modulation techniques. We illustrate this in the following for MPSK and MQAM.

For MPSK,

$$P_{e,\mathrm{MPSK}} = \sum_{l=1}^{L} \sum_{k=1}^{\mu_l/2} A_{l,k} \; P_{e,\mathrm{MPSK}}(k, 2k\lambda_l)\,. \qquad (22)$$

The elementary SEP for single-branch reception of MPSK in Nakagami fading with parameter pair (m_1, Γ_1) is given by

$$P_{e,\mathrm{MPSK}}(m_1, \Gamma_1) = \frac{1}{\pi} \int_0^{\Theta} \left[\frac{\sin^2\theta}{c_{\mathrm{MPSK}} \frac{\Gamma_1}{m_1} + \sin^2\theta} \right]^{m_1} d\theta\,, \qquad (23)$$

where $c_{\mathrm{MPSK}} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. Note that (23) is equivalent to the SEP for reception of MPSK over Rayleigh-fading channels using MRC with m_1 *identical* branches having *equal* SNR's of $\frac{\Gamma_1}{m_1}$, and therefore a closed-form expression for (23) can be found in [27].

For MQAM,

$$P_{e,\mathrm{MQAM}} = \sum_{l=1}^{L} \sum_{k=1}^{\mu_l/2} A_{l,k} \; P_{e,\mathrm{MQAM}}(m_1, 2k\lambda_l)\,. \qquad (24)$$

The elementary SEP for single-branch reception of MQAM in Nakagami fading with parameter pair (m_1, Γ_1) is

$$\begin{aligned} P_{e,\mathrm{MQAM}}(m_1, \Gamma_1) &= q\,\frac{1}{\pi} \int_0^{\pi/2} \left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}} \frac{\Gamma_1}{m_1} + \sin^2\theta} \right]^{m_1} d\theta \\ &- \frac{q^2}{4}\,\frac{1}{\pi} \int_0^{\pi/4} \left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}} \frac{\Gamma_1}{m_1} + \sin^2\theta} \right]^{m_1} d\theta\,, \qquad (25) \end{aligned}$$

where $q = 4(1 - \frac{1}{\sqrt{M}})$, and $c_{\mathrm{MQAM}} = \frac{3}{2(M-1)}$. Note again that (25) is equivalent to the SEP for reception of MQAM over Rayleigh-fading channels using MRC with m_1 *identical* branches having *equal* SNR's of $\frac{\Gamma_1}{m_1}$ and therefore a closed-form expression for (25) can be found in [28].

C. Canonical Form for Outage Probability

Similar to the derivation for the canonical structure for SEP derived in section III-B, the canonical structure for outage probability is derived in this section using the p.d.f. of the combiner output SNR given in (18).

The outage probability, $P_{\mathrm{out}}(x)$, is defined as the probability that the instantaneous SNR falls below a threshold x [9]. Mathematically,

$$P_{\mathrm{out}}(x) = \Pr\{\gamma_{\mathrm{MRC}} \leq x\}\,. \qquad (26)$$

This can be obtained by direct integration of the p.d.f. of the combiner output SNR of (18) as,

$$\begin{aligned} P_{\mathrm{out}}(x) &= \int_0^x f_{\gamma_{\mathrm{MRC}}}(\gamma)\, d\gamma \\ &= \sum_{l=1}^{L} \sum_{k=1}^{\mu_l/2} A_{l,k} \int_0^x g_\gamma(\gamma; \frac{1}{2\lambda_l}, k)\, d\gamma\,. \qquad (27) \end{aligned}$$

Let the outage probability for a single branch with p.d.f. $g_\gamma(\gamma; \frac{1}{2\lambda_l}, k)$ be denoted by $P_{\mathrm{out}}(x; k, 2k\lambda_l)$. Recognizing that the integral in (27) is equal to $P_{\mathrm{out}}(x; k, 2k\lambda_l)$,

$$P_{\mathrm{out}}(x) = \sum_{l=1}^{L} \sum_{k=1}^{\mu_l/2} A_{l,k} \; P_{\mathrm{out}}(x; k, 2k\lambda_l)\,. \qquad (28)$$

In other words, the outage probability for MRC with N branches, where m_i and Γ_i are not necessarily equal among the branches, in correlated Nakagami fading is simply the weighted sum of the "elementary outage probabilities." The elementary terms are the outage probability of the single-branch receiver in Nakagami fading with parameter pair $(k, 2k\lambda_l)$, or equivalently the outage probability of a k-branch receiver in Rayleigh-fading channels using MRC with *equal* SNR's of $2k\lambda_l$.

IV. Limiting Cases

In this section, we study some limiting cases of the results obtained in Section III to verify that our results agree with previously-published results.

A. Single-branch Nakagami channel

Let us first consider the simplest possible case, namely single-branch reception (without diversity) in Nakagami fading with (m_1, Γ_1). The p.d.f., SEP, and outage probability for this case are also the elementary expressions in the canonical structure. For single-branch reception ($N = 1$), K_X is a $2m_1 \times 2m_1$ diagonal matrix having only one eigenvalue ($L = 1$) of multiplicity $\mu_1 = 2m_1$. The eigenvalue is $\lambda_1 = \frac{\Gamma_1}{2m_1}$. Substituting the appropriate parameters into (18) results in

$$f_\gamma(\gamma) = g_\gamma(\gamma; \frac{m_1}{\Gamma_1}, m_1)\,, \qquad (29)$$

which is simply the Nakagami p.d.f. with (m_1, Γ_1), as it should be. Similarly, the SEP for (21) reduces to

$$P_e = P_e(m_1, \Gamma_1), \qquad (30)$$

where the closed-form expression of the $P_e(m_1, \Gamma_1)$ for binary frequency shift keying (FSK) is given in [6], for BFSK and BPSK is given in [7], [12], and for MFSK is given in [8]. Alternatively, expressions for the $P_e(m_1, \Gamma_1)$ for MPSK and MQAM, in the form of a single integral with finite limits with the integrand involving trigonometric functions that can be evaluated numerically, are given in [29]. Similarly, (28) reduces to

$$P_{\text{out}}(x) = P_{\text{out}}(x; m_1, \Gamma_1) . \tag{31}$$

B. Independent Nakagami Channels

B.1 Case 1: *All* Equal $\frac{\Gamma_i}{m_i}$

Here, we consider MRC with independent Nakagami fading with parameter pairs (m_i, Γ_i) such that $\frac{\Gamma_i}{m_i}$ are equal for *all* N branches. Note that this includes the case of $m_i = m$ and $\Gamma_i = \Gamma$. In this case, K_X is a $2mN \times 2mN$ diagonal matrix having only one eigenvalue ($L = 1$) of multiplicity $\mu_1 = 2mN$. The eigenvalue is $\lambda_1 = \frac{\Gamma}{2m}$. Substituting the appropriate parameters into (18) results in

$$f_\gamma(\gamma) = g_\gamma(\gamma; \frac{m}{\Gamma}, Nm) , \tag{32}$$

which is simply the Nakagami p.d.f. with $(Nm, N\Gamma)$ as in [6]. Similarly, the SEP for (21) reduces to

$$P_e = P_e(Nm, N\Gamma), \tag{33}$$

which agrees with the results for BFSK [6], and the results for BFSK and BPSK given in [12]. Similarly, (28) reduces to

$$P_{\text{out}}(x) = P_{\text{out}}(x; Nm, N\Gamma) . \tag{34}$$

B.2 Case 2: *All* Distinct $\frac{\Gamma_i}{m_i}$

Here, we consider MRC with independent Nakagami fading with parameter pairs (m_i, Γ_i) such that $\frac{\Gamma_i}{m_i}$ are distinct for *all* N branches. Note that this includes the case for equal m_i but distinct Γ_i.[6] In this case, K_X has N distinct eigenvalues given by $\lambda_l = \frac{\Gamma_l}{2m_l}$, $l = 1, \ldots, N$, each with multiplicity $\mu_l = 2m_l$. Substituting the appropriate parameters into (18) results in

$$f_{\gamma_{\text{MRC}}}(\gamma) = \sum_{l=1}^{N} \sum_{k=1}^{m_l} A_{l,k} \, g_\gamma(\gamma; \frac{m_l}{\Gamma_l}, k) , \tag{35}$$

which agrees with [6]. Similarly, the SEP for (21) reduces to

$$P_e = \sum_{l=1}^{N} \sum_{k=1}^{m_l} A_{l,k} \, P_e(k, k\frac{\Gamma_l}{m_l}) , \tag{36}$$

which agrees with the results for BFSK given in [6]. Similarly, (28) reduces to

$$P_{\text{out}}(x) = \sum_{l=1}^{M} \sum_{k=1}^{m_l} A_{l,k} \, P_{\text{out}}(x; k, k\frac{\Gamma_l}{m_l}) . \tag{37}$$

[6]This case was analyzed in [11] by approximating the sum of the squares of Nakagami r.v.'s by a single Nakagami r.v. with appropriate parameters. This approximation becomes exact when the Γ_i's are all equal.

C. Identical Nakagami Channels with Correlation

Consider N branch MRC in Nakagami fading, with arbitrary correlation where the m_i's and Γ_i's are equal for all branches. This includes the two special cases considered in [12] with two specific correlation models, namely equal correlation and exponential correlation. In this case, the $\{\lambda_l\}$'s are the L *distinct* eigenvalues of the block-diagonal covariance matrix having m identical $N \times N$ diagonal blocks given by

$$K_{\tilde{Y}_i} = \frac{\Gamma}{2m} \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{1,2} & 1 & \cdots & \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,N} & \rho_{2,N} & \cdots & 1 \end{bmatrix} . \tag{38}$$

The p.d.f., SEP, and outage probability can be obtained by substituting the eigenvalues into (18), (21), and (28), respectively. Let $\{\tilde{\lambda}_l\}_{l=1}^N$ be *all* of the N eigenvalues of $K_{\tilde{Y}_i}$. Then the c.f. of the N-branch diversity combiner output SNR can be shown, using (16), to be

$$\begin{aligned} \psi_{\gamma_{\text{MRC}}}(j\nu) &= \prod_{l=1}^{N} \left[1 - j\nu 2\tilde{\lambda}_l\right]^{-m} \\ &= |I - j\nu 2\Lambda|^{-m} , \end{aligned} \tag{39}$$

where Λ is an $N \times N$ diagonal matrix where the diagonal components are the elements $\{\tilde{\lambda}_l\}$.[7] Since $K_{\tilde{Y}_i}$, is Hermitian symmetric, there exists an *unitary* matrix E with $EE^t = I$ such that $K_{\tilde{Y}_i} = E\Lambda E^t$ [30]. Therefore (39) becomes

$$\begin{aligned} \psi_{\gamma_{\text{MRC}}}(j\nu) &= \left|EE^t - j\nu 2E\Lambda E^t\right|^{-m} \\ &= \left|I - j\nu 2K_{\tilde{Y}_i}\right|^{-m} . \end{aligned} \tag{40}$$

This agrees with a Model C of [31].

V. Conclusions

We developed an analytical framework to study the performance of wireless systems using maximal ratio combining with an arbitrary number of diversity branches in correlated fading. We considered coherent detection of digital signals received over correlated Nakagami fading channels, where the instantaneous SNR's of the diversity branches are *not* necessarily independent *or* identically distributed.

We derived closed-form expressions for three performance measures of an MRC diversity system: 1) probability density function (p.d.f.) of the combiner output SNR; 2) symbol error probability (SEP) with coherent detection; and 3) outage probability. We obtained a *canonical structure* for these performance measures as a weighted sum of the corresponding expressions for a non-diversity (single-branch) system with appropriately-defined parameters. This result is fundamental: the canonical structure

[7]In general, the elements of $\{\tilde{\lambda}_l\}$ are not necessarily distinct and hence $\{\tilde{\lambda}_l\}_{l=1}^N \supseteq \{\lambda_l\}_{l=1}^L$. If the elements of $\{\tilde{\lambda}_l\}$ are not distinct, $\{\lambda_l\}$ is a proper subset of $\{\tilde{\lambda}_l\}$ with $L < N$. When the $\{\tilde{\lambda}_l\}$ are all distinct, then set equality is achieved, i.e., $\{\tilde{\lambda}_l\} = \{\lambda_l\}$ with $L = N$.

depends *only* on the properties of the channel and diversity combiner, and *not* on the specific modulation technique.

Thus, lengthy derivations are no longer needed for separate cases of wireless scenarios with different numbers of diversity branches, modulation techniques, and correlation models, as our results give a simple prescription for computing the parameters of single-branch Nakagami channels, the weights, and the number of terms used in the sum to calculate the results. Calculations of SEP for specific modulation techniques were illustrated through examples. Our results extend previously-derived results to cover numerous additional useful cases.

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