Exact Error Probability Expressions for H-S/MRC in Rayleigh Fading: A Virtual Branch Technique

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Abstract— **In this paper, we use a virtual branch technique to derive *exact* expressions for the symbol error probability (SEP) of hybrid selection/maximal-ratio combining (H-S/MRC) in a multipath fading environment. We consider coherent detection of M-ary phase-shift keying and quadrature amplitude modulation for the case of independent Rayleigh fading with equal signal-to-noise ratio averaged over the fading on each diversity branch. We further obtain a *canonical structure* for the SEP of H-S/MRC as a weighted sum of the elementary SEP's, which are the SEP's for MRC of independent and identically distributed branches, whose closed-form expressions are well-known.**

I. Introduction

Hybrid selection/maximal-ratio combining (H-S/MRC) (see for example [1], [2], [3]) is a reduced complexity diversity combining scheme, where L (with the largest signal-to-noise ratio (SNR) at each instant) out of N diversity branches are selected and combined using maximal-ratio combining (MRC). This technique provides improved performance over L branch MRC when additional diversity is available, without requiring additional electronics and/or power consumption.

In previous papers [2], [3], we derived the mean and variance of the combiner output SNR of H-S/MRC for any L and N under the assumption of independent Rayleigh fading on each diversity branch with equal SNR averaged over the fading. This was performed using a "virtual branch" technique which simplified the derivations.[1]

In this paper we extend [2], [3] to derive analytical expressions for the SEP with H-S/MRC. The proposed problem is made analytically tractable by: 1) transforming the physical diversity branches into the "virtual branch" domain as in [2], [3]; and 2) using alternative definite integral representations of the conditional SEP's with finite limits for MPSK and MQAM; which results in a simple derivation of the SEP for *arbitrary* L and N. We further obtain a *canonical structure* for the SEP of H-S/MRC as a weighted sum of the elementary SEP's, which are the SEP's for MRC of independent identically distributed (i.i.d.) branches, whose closed-form expressions are well-known.

In Section II, we derive the SEP for general diversity combining and apply these results to H-S/MRC in Section III. The canonical forms for SEP's are derived in Section IV. In Section V, we present numerical examples. Conclusions are given in Section VI.

II. Diversity Combining Analysis

A. Virtual Branch Technique: The Key Idea

The analysis of H-S/MRC based on a chosen ordering of the branches at first appears to be complicated, since the SNR statistics of the ordered branches are *not* independent. Here, we alleviate this problem by transforming the ordered-branch variables into a new set of i.i.d. *virtual branches*, and expressing the ordered-branch SNR variables as a linear function of i.i.d. virtual branch SNR variables. The key advantage of this formulation is that it allows greater flexibility in the selection process of the ordered instantaneous SNR values, and permits the combiner output SNR to be expressed in terms of the i.i.d. virtual branch SNR variables. In this framework, the derivation of the SEP of H-S/MRC, involving the evaluation of nested N-fold integrals, essentially reduces to the evaluation of a single integral. The well-known results for selection diversity (SD) and MRC are shown to be special cases of our results.

B. General Theory

Let γ_i denote the instantaneous SNR of the $i^{\underline{\text{th}}}$ diversity branch defined by $\gamma_i \triangleq \alpha_i^2 E_s/N_{0i}$, where E_s is the average symbol energy, and α_i is the instantaneous fading amplitude and N_{0i} is the noise power spectral density of the $i^{\underline{\text{th}}}$ branch. We model the α_i's as i.i.d. Rayleigh random variables (r.v.'s) and thus γ_i's are i.i.d. continuous r.v.'s with exponential p.d.f. and mean $\Gamma = \mathbb{E}\{\gamma_1\}$.

Let us first consider a general diversity combining (GDC)

[1] We also used the virtual branch technique in [4] to derive the symbol error probability (SEP) for coherent detection of M-ary phase-shift keying (MPSK) and quadrature amplitude modulation (MQAM) using MRC with an arbitrary number of diversity branches in correlated Nakagami fading channels, where the instantaneous SNR's of the diversity branches are *not* necessarily independent *or* identically distributed. In [5], we extended the results of [4] and obtained canonical structures for the probability density function (p.d.f.) of the combiner output SNR, SEP with coherent detection, and outage probability.

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with the instantaneous output SNR of the form

$$\gamma_{\rm GDC} = \sum_{i=1}^{N} a_i \gamma_{(i)} \,, \tag{1}$$

where $a_i \in \{0,1\}$, $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \ldots > \gamma_{(N)}$, and N is the number of available diversity branches. It will be apparent later that several diversity combining schemes, including H-S/MRC, turn out to be special cases of (1). Note that the possibility of at least two equal $\gamma_{(i)}$'s is excluded, since $\gamma_{(i)} \neq \gamma_{(j)}$ *almost surely* for continuous random variables.[2] Denoting $\boldsymbol{\gamma}_{(N)} \triangleq (\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)})$, the joint p.d.f. of $\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)}$ is [2], [3]

$$f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N}) = \begin{cases} \frac{N!}{\Gamma^N} e^{-\frac{1}{\Gamma}\sum_{m=1}^{N}\gamma_{(m)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

It is important to note that the $\gamma_{(i)}$'s are *no* longer independent, even though the underlying γ_i's are independent.

C. *Symbol Error Probability for GDC over the Channel Ensemble*

The SEP for GDC in multipath-fading environment is obtained by averaging the conditional SEP over the channel ensemble. This can be accomplished by averaging the $\Pr\{e|\gamma_{\rm GDC}\}$ over the p.d.f. of the $\gamma_{\rm GDC}$ as

$$\begin{aligned} P_{e,\rm GDC} &= \mathbb{E}_{\gamma_{\rm GDC}}\{\Pr\{e|\gamma_{\rm GDC}\}\} \\ &= \int_0^\infty \Pr\{e|\gamma\}\, f_{\gamma_{\rm GDC}}(\gamma) d\gamma \,, \end{aligned} \tag{3}$$

where $\Pr\{e|\gamma_{\rm GDC}\}$ is the *conditional* SEP, conditioned on the random variable $\gamma_{\rm GDC}$, and $f_{\gamma_{\rm GDC}}(\cdot)$ is the p.d.f. of the combiner output SNR [8], [9]. Alternatively, averaging over channel ensemble can be accomplished, using the technique of [10], [11], by substituting the expression for $\gamma_{\rm GDC}$ directly in terms of the physical branch variables given in (1), as

$$\begin{aligned} P_{e,\rm GDC} &= \mathbb{E}_{\{\gamma_{(i)}\}}\Big\{\Pr\Big\{e|\gamma_{\rm GDC} = \textstyle\sum_{i=1}^N a_i\gamma_{(i)}\Big\}\Big\} \\ &= \int_0^\infty\int_0^{\gamma_{(1)}}\cdots\int_0^{\gamma_{(N-1)}} \Pr\Big\{e|\textstyle\sum_{i=1}^N a_i\gamma_{(i)}\Big\} \\ &\times \quad f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^N)\, d\gamma_{(N)}\ldots d\gamma_{(2)} d\gamma_{(1)} \,. \end{aligned} \tag{4}$$

Since the statistics of the ordered-branches are *no* longer independent, the evaluation of (4) involves nested N-fold integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR of the ordered diversity branches, $\gamma_{(i)}$, into a new set of *virtual branch* instantaneous SNR's, V_n's, using the following relation:

$$\gamma_{(i)} = \sum_{n=i}^{N} \frac{\Gamma}{n} V_n \,. \tag{5}$$

[2]In our context, the notion of "almost surely" or "almost everywhere" can be stated mathematically as: if $\mathcal{N} = \{\gamma_{(i)} = \gamma_{(j)}\}$, then $\Pr\{\mathcal{N}\} = 0$ [6], [7].

It can be verified that the instantaneous SNR's of the virtual branches are i.i.d. normalized exponential random variables with characteristic function given by

$$\psi_{V_n}(j\nu) \triangleq \mathbb{E}\left\{e^{+j\nu V_n}\right\} = \frac{1}{1-j\nu} \,. \tag{6}$$

The instantaneous SNR of the combiner output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{\rm GDC} = \sum_{n=1}^{N} b_n V_n \,, \tag{7}$$

where the coefficients b_n are given by

$$b_n = \frac{\Gamma}{n} \sum_{i=1}^{n} a_i \,. \tag{8}$$

Using the *independent* virtual branches, the N-fold nested integrals of (4) reduce to

$$\begin{aligned} P_{e,\rm GDC} &= \mathbb{E}_{\{V_n\}}\Big\{\Pr\Big\{e|\gamma_{\rm GDC} = \textstyle\sum_{n=1}^N b_n V_n\Big\}\Big\} \\ &= \int_0^\infty\int_0^\infty\cdots\int_0^\infty \Pr\Big\{e|\textstyle\sum_{n=1}^N b_n v_n\Big\} \displaystyle\prod_{n=1}^N f_{V_n}(v_n) dv_n \,. \end{aligned} \tag{9}$$

For many important modulation techniques, it can be shown that $\Pr\left\{e|\sum_{n=1}^N b_n V_n\right\}$ factors into a product of N terms, where each term depends *only* on one of the V_n's. We will illustrate this by the following two important examples.

C.1 SEP for MPSK with GDC

For coherent detection of MPSK, an alternative representation $\Pr\{e|\gamma_{\rm GDC}\}$, involving a definite integral with *finite* limits, is given by [12], [13]

$$\Pr\{e_{\rm MPSK}|\gamma_{\rm GDC}\} = \frac{1}{\pi}\int_0^\Theta e^{-\frac{c_{\rm MPSK}}{\sin^2\theta}\gamma_{\rm GDC}} d\theta \,, \tag{10}$$

where $c_{\rm MPSK} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. Substituting (10) into (3), the SEP for MPSK becomes

$$\begin{aligned} P_{e,\rm GDC}^{\rm MPSK} &= \frac{1}{\pi}\int_0^\Theta \mathbb{E}_{\gamma_{\rm GDC}}\Big\{e^{-\frac{c_{\rm MPSK}}{\sin^2\theta}\gamma_{\rm GDC}}\Big\}\, d\theta \\ &= \frac{1}{\pi}\int_0^\Theta \int_0^\infty e^{-\frac{c_{\rm MPSK}}{\sin^2\theta}\gamma} f_{\gamma_{\rm GDC}}(\gamma) d\gamma\, d\theta \,. \end{aligned} \tag{11}$$

Although, the evaluation of (11) involves a single integration for averaging over the channel ensemble, it requires knowledge of the p.d.f. of $\gamma_{\rm GDC}$. Alternatively, we substitute (10) into (4), and the SEP for MPSK becomes

$$\begin{aligned} P_{e,\rm GDC}^{\rm MPSK} &= \frac{1}{\pi}\int_0^\Theta \mathbb{E}_{\{\gamma_{(i)}\}}\Big\{e^{-\frac{c_{\rm MPSK}}{\sin^2\theta}\sum_{i=1}^N a_i\gamma_{(i)}}\Big\}\, d\theta \\ &= \frac{1}{\pi}\int_0^\Theta \int_0^\infty\int_0^{\gamma_{(1)}}\cdots\int_0^{\gamma_{(N-1)}} e^{-\frac{c_{\rm MPSK}}{\sin^2\theta}\sum_{i=1}^N a_i\gamma_{(i)}} \\ &\times \quad f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^N)\, d\gamma_{(N)}\ldots d\gamma_{(2)} d\gamma_{(1)}\, d\theta \,. \end{aligned} \tag{12}$$

Note in (12) that, since the ordered physical branches are *no* longer independent, direct use of the methods given in [10], [11] requires an N-fold nested integration for the expectation operation in (12). This is alleviated using the virtual branch technique by substituting (10) into (9) as:

$$\begin{aligned} P_{e,\text{GDC}}^{\text{MPSK}} &= \frac{1}{\pi}\int_0^{\Theta} \mathbb{E}_{\{V_n\}}\Big\{e^{-\frac{c_{\text{MPSK}}}{\sin^2\theta}\sum_{n=1}^{N} b_n V_n}\Big\}\, d\theta \\ &= \frac{1}{\pi}\int_0^{\Theta} \prod_{n=1}^{N} \psi_{V_n}\left(-\frac{c_{\text{MPSK}} b_n}{\sin^2\theta}\right) d\theta\,, \end{aligned} \tag{13}$$

where we have used the fact that V_n's are independent. The usefulness of the virtual path technique is apparent by observing that the expectation operation in the above equation no longer requires an N-fold nested integration.

Substituting (6) into (13) gives

$$P_{e,\text{GDC}}^{\text{MPSK}} = \frac{1}{\pi}\int_0^{\Theta} \prod_{n=1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MPSK}} b_n + \sin^2\theta}\right] d\theta\,. \tag{14}$$

Thus the derivation of the SEP for coherent detection of MPSK using N-branch GDC, involving the N-fold nested integrals in (12), essentially reduces to a single integral over θ with finite limits. The integrand is an N-fold product of a simple expression involving trigonometric functions. Note that the independence of the virtual branch variables plays a key role in simplifying the derivation.

D. SEP for MQAM with GDC

For coherent detection of MQAM with $M = 2^k$ for even k, $\Pr\{e|\gamma_{\text{GDC}}\}$ is given by [11]

$$\begin{aligned} \Pr\{e_{\text{MQAM}}|\gamma_{\text{GDC}}\} &= q\,\frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{c_{\text{MQAM}}}{\sin^2\theta}\gamma_{\text{GDC}}} d\theta \\ &\quad - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4} e^{-\frac{c_{\text{MQAM}}}{\sin^2\theta}\gamma_{\text{GDC}}} d\theta\,, \end{aligned} \tag{15}$$

where $q = 4(1-\frac{1}{\sqrt{M}})$, and $c_{\text{MQAM}} = \frac{3}{2(M-1)}$. Using the virtual branch technique, similar to the steps used for MPSK in Section II-C.1, the SEP for MQAM becomes

$$\begin{aligned} P_{e,\text{GDC}}^{\text{MQAM}} &= q\,\frac{1}{\pi}\int_0^{\pi/2} \prod_{n=1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MQAM}} b_n + \sin^2\theta}\right] d\theta \\ &\quad - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4} \prod_{n=1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MQAM}} b_n + \sin^2\theta}\right] d\theta\,. \end{aligned} \tag{16}$$

Again, the derivation of the SEP for coherent detection of MQAM using GDC in Rayleigh fading reduces to two terms, each consisting of a single integral over θ involving trigonometric functions with finite limits.

III. Application of General Theory

The results given in (14) and (16) are for the SEP for coherent detection of MPSK and MQAM, respectively, using N-branch GDC in Rayleigh-fading channels. The b_n's in (14) and (16) depend on the choice of a_i's via (8). The results obtained in (14) and (16) are *general* in the sense that they apply to a variety of diversity combining systems that fit into the form of (1), including H-S/MRC, SD, and MRC.

A. SEP with H-S/MRC

In this section, the general theory derived in Section II is used to evaluate the performance of H-S/MRC. The instantaneous output SNR of H-S/MRC is

$$\gamma_{\text{H-S/MRC}} = \sum_{i=1}^{L} \gamma_{(i)}\,, \tag{17}$$

where $1 \le L \le N$. Note that $\gamma_{\text{H-S/MRC}} = \gamma_{\text{GDC}}$ with

$$a_i = \begin{cases} 1, & i = 1, \ldots, L \\ 0, & \text{otherwise}. \end{cases} \tag{18}$$

In this case,

$$b_n = \begin{cases} \Gamma, & n \le L \\ \Gamma\frac{L}{n}, & \text{otherwise}. \end{cases} \tag{19}$$

Substituting (19) into (14), the SEP for MPSK with H-S/MRC can be easily obtained as

$$\begin{aligned} P_{e,\text{H-S/MRC}}^{\text{MPSK}} &= \frac{1}{\pi}\int_0^{\Theta} \left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma + \sin^2\theta}\right]^L \\ &\quad \times \prod_{n=L+1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma\frac{L}{n} + \sin^2\theta}\right] d\theta\,. \end{aligned} \tag{20}$$

Similarly, the SEP for MQAM with H-S/MRC can be obtained by substituting (19) into (16) as

$$\begin{aligned} P_{e,\text{H-S/MRC}}^{\text{MQAM}} &= q\,\frac{1}{\pi}\int_0^{\pi/2} \left[\frac{\sin^2\theta}{c_{\text{MQAM}}\Gamma + \sin^2\theta}\right]^L \\ &\quad \times \prod_{n=L+1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MQAM}}\Gamma\frac{L}{n} + \sin^2\theta}\right] d\theta \\ &\quad - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4} \left[\frac{\sin^2\theta}{c_{\text{MQAM}}\Gamma + \sin^2\theta}\right]^L \\ &\quad \times \prod_{n=L+1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MQAM}}\Gamma\frac{L}{n} + \sin^2\theta}\right] d\theta\,. \end{aligned} \tag{21}$$

B. Special Case 1: SD

SD is the simplest form of diversity combining whereby the received signal from *one* of N diversity branches is selected [14]. The output SNR of SD is

$$\gamma_{\text{SD}} = \max_i \{\gamma_i\} = \gamma_{(1)}\,. \tag{22}$$

Note that SD is a special case of H-S/MRC with $L = 1$. Substituting $L = 1$ into (20) and (21), the SEP for MPSK and MQAM with SD becomes

$$P_{e,\text{SD}}^{\text{MPSK}} = \frac{1}{\pi}\int_0^{\Theta} \prod_{n=1}^{N} \left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma\frac{1}{n} + \sin^2\theta}\right] d\theta\,, \tag{23}$$

and

$$P_{e,\mathrm{SD}}^{\mathrm{MQAM}} = q\,\frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\Gamma\frac{1}{n}+\sin^2\theta}\right]d\theta - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\prod_{n=1}^{N}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\Gamma\frac{1}{n}+\sin^2\theta}\right]d\theta\,, \quad (24)$$

respectively.

C. Special Case 2: MRC

In MRC, the received signals from *all* diversity branches are weighted and combined to maximize the SNR at the combiner output [14]. The output SNR of MRC is

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{N}\gamma_i = \sum_{i=1}^{N}\gamma_{(i)}\,. \quad (25)$$

Since MRC is a special case of H-S/MRC with $L = N$, the SEP expressions for MPSK and MQAM with MRC are obtained by setting $L = N$ in (20) and (21) as

$$P_{e,\mathrm{MRC}}^{\mathrm{MPSK}} = \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{c_{\mathrm{MPSK}}\Gamma+\sin^2\theta}\right]^N d\theta \triangleq P_{e,\mathrm{MRC}}^{\mathrm{MPSK}}(N,\Gamma)\,, \quad (26)$$

and

$$P_{e,\mathrm{MRC}}^{\mathrm{MQAM}} = q\,\frac{1}{\pi}\int_0^{\pi/2}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\Gamma+\sin^2\theta}\right]^N d\theta - \frac{q^2}{4}\frac{1}{\pi}\int_0^{\pi/4}\left[\frac{\sin^2\theta}{c_{\mathrm{MQAM}}\Gamma+\sin^2\theta}\right]^N d\theta \triangleq P_{e,\mathrm{MRC}}^{\mathrm{MQAM}}(N,\Gamma)\,. \quad (27)$$

Specifically (26) and (27) are the SEP for coherent detection of MPSK and MQAM using MRC with N *independent* branches having *equal* average SNR's of Γ in Rayleigh fading, and therefore a closed-form expressions for (26) and (27) can be found in [15] and [16] respectively. Note that (26) and (27) are equivalent to the SEP for *single*-branch reception of MPSK and MQAM in Nakagami-m fading with fading parameter N (i.e., $m = N$) having an average SNR of $N\Gamma$ [4], [5].[3]

IV. Canonical Form for SEP's

A. Canonical Form for SEP's with GDC

The quest for obtaining insights from (14) and (16) is at its peak, which leads to an expansion for the integrand in (14). Let $\{\tilde{b}_n\}$ be the set of $\tilde{N}$ *distinct* values of $\{b_n\}$ where each $\tilde{b}_n$ has algebraic multiplicity μ_n such that $\sum_{n=1}^{\tilde{N}}\mu_n = N$. Then (14) can be rewritten as

$$P_{e,\mathrm{GDC}}^{\mathrm{MPSK}} = \sum_{n=1}^{\tilde{N}}\sum_{k=1}^{\mu_n} A_{n,k}\,\frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{c_{\mathrm{MPSK}}\tilde{b}_n+\sin^2\theta}\right]^k d\theta\,, \quad (28)$$

[3]The fading parameter of Nakagami fading, usually denoted by the symbol m, is also known as fade parameter, fading severity factor, fading figure, or (inverse) fading-depth parameter.

where the $A_{n,k}$'s are weighting coefficients of the expansion. Comparing (28) with (26)

$$P_{e,\mathrm{GDC}}^{\mathrm{MPSK}} = \sum_{n=1}^{\tilde{N}}\sum_{k=1}^{\mu_n} A_{n,k}\,P_{e,\mathrm{MRC}}^{\mathrm{MPSK}}(k,\tilde{b}_n)\,. \quad (29)$$

Interesting insights can now be obtained from (29). The SEP for MPSK using N-branch GDC in Rayleigh fading is simply the weighted sum of the elementary SEP's. The elementary SEP's for the (n,k)-entries are simply the SEP's for the coherent detection of MPSK using MRC with k *independent* branches having *equal* SNR's of $\tilde{b}_n$ in Rayleigh-fading, or equivalently the SEP for *single*-branch reception of MPSK in Nakagami fading with fading parameter equal to k having an average SNR of $k\,\tilde{b}_n$.

Similar structure, namely, linear combination of the simple "elementary SEP's," for MQAM can also be obtained from (16) as

$$P_{e,\mathrm{GDC}}^{\mathrm{MQAM}} = \sum_{n=1}^{\tilde{N}}\sum_{k=1}^{\mu_n} A_{n,k}\,P_{e,\mathrm{MRC}}^{\mathrm{MQAM}}(k,\tilde{b}_n)\,. \quad (30)$$

B. Canonical Forms for SEP with H-S/MRC

For H-S/MRC, it can be seen from (19) that the number of distinct values of $\{b_n\}$ is $\tilde{N} = N - L + 1$. The distinct values of $\tilde{b}_n$'s are given by

$$\tilde{b}_n = \begin{cases}\Gamma, & n = 1\\ \Gamma\frac{L}{L+n-1}, & n = 2,\ldots,\tilde{N}\,,\end{cases} \quad (31)$$

and their multiplicities μ_n's are given by

$$\mu_n = \begin{cases} L, & n = 1\\ 1, & n = 2,\ldots,\tilde{N}\,.\end{cases} \quad (32)$$

Substituting (31) and (32) into (29) and (30), we arrive at the canonical forms for the SEP of MPSK and MQAM with H-S/MRC, which are given respectively by

$$P_{e,\mathrm{H\text{-}S/MRC}}^{\mathrm{MPSK}} = \sum_{k=1}^{L} A_{1,k}\,P_{e,\mathrm{MRC}}^{\mathrm{MPSK}}(k,\Gamma) + \sum_{n=2}^{N-L+1} A_{n,1}\,P_{e,\mathrm{MRC}}^{\mathrm{MPSK}}\left(1,\Gamma\frac{L}{L+n-1}\right), \quad (33)$$

and

$$P_{e,\mathrm{H\text{-}S/MRC}}^{\mathrm{MQAM}} = \sum_{k=1}^{L} A_{1,k}\,P_{e,\mathrm{MRC}}^{\mathrm{MQAM}}(k,\Gamma) + \sum_{n=2}^{N-L+1} A_{n,1}\,P_{e,\mathrm{MRC}}^{\mathrm{MQAM}}\left(1,\Gamma\frac{L}{L+n-1}\right). \quad (34)$$

C. Canonical SEP Forms with SD

Recall that SD is a special case of H-S/MRC with $L = 1$. Thus the canonical forms for SEP of MPSK and MQAM with SD are obtained by setting $L = 1$ in (33) and (34) as

$$P_{e,\mathrm{SD}}^{\mathrm{MPSK}} = \sum_{n=1}^{N} A_{n,1}\,P_{e,\mathrm{MRC}}^{\mathrm{MPSK}}\left(1,\frac{\Gamma}{n}\right), \quad (35)$$

and

$$P_{e,\mathrm{SD}}^{\mathrm{MQAM}} = \sum_{n=1}^{N} A_{n,1}\, P_{e,\mathrm{MRC}}^{\mathrm{MQAM}}(1, \frac{\Gamma}{n}) . \qquad (36)$$

V. Numerical Examples

In this section, the results derived in the previous section for H-S/MRC are illustrated. The notation H-L/N is used to denote H-S/MRC that selects and combines L out of N branches. Note that H-1/1 is a single branch receiver, and H-1/N and H-N/N are N-branch SD and MRC, respectively.

Figures 1 and 2 show the SEP for coherent detection of MPSK with M=8 (8-PSK) versus average SNR per branch for various L with $N = 4$ and $N = 8$, respectively. Note that SD and MRC upper and lower bound, respectively, the SEP for H-S/MRC. It is seen that most of the gain of H-S/MRC is achieved for small L, e.g., H-S/MRC is within 1.1 dB of MRC when $L = N/2$.

Figures 3 and 4 show the SEP for coherent detection of 8-PSK versus average SNR per branch for various N with $L = 2$ and $L = 4$, respectively. Note that, although the incremental gain with each additional antenna becomes smaller as N increases, the gain with each additional antenna is still significant even with $N = 8$. Furthermore, for $L = 2$ and $L = 4$, at a 10^{-4} SEP, H-S/MRC with $N = 8$ requires 12.5 and 4.5 dB, respectively, lower SNR than L-branch MRC.

VI. Conclusions

We derived exact SEP expressions for coherent detection of MPSK and MQAM with H-S/MRC in multipath-fading wireless environments. We analyzed this system using a "virtual branch" technique which resulted in a simple derivation of the SEP for *arbitrary* L and N. The key idea was to transform the dependent ordered-branch variables into a new set of i.i.d. *virtual branches*, and express the combiner output SNR as a linear combination of the i.i.d. virtual branch SNR variables. We further obtained a *canonical structure* for the SEP of H-S/MRC as a weighted sum of the elementary SEP's, which are the SEP's for MRC of independent and identically distributed branches, whose closed-form expressions are well-known.

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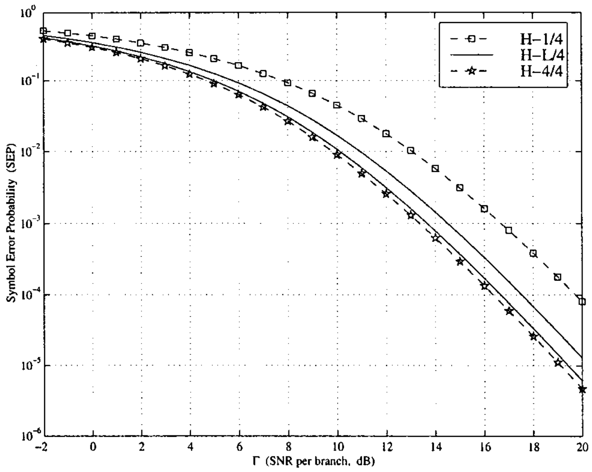

Fig. 1. The symbol error probability for coherent detection of 8-PSK with H-S/MRC as a function of average SNR per branch for various L with $N = 4$. The curves are parameterized by different H-L/N, starting from the highest curve representing H-1/4, and decrease monotonically to the lowest curve representing H-4/4.

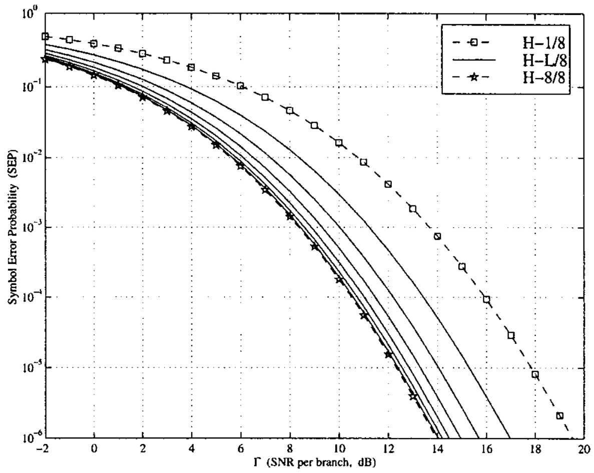

Fig. 2. The symbol error probability for coherent detection of 8-PSK with H-S/MRC as a function of average SNR per branch for various L with $N = 8$. The curves are parameterized by different H-L/N, starting from the highest curve representing H-1/8, and decrease monotonically to the lowest curve representing H-8/8.

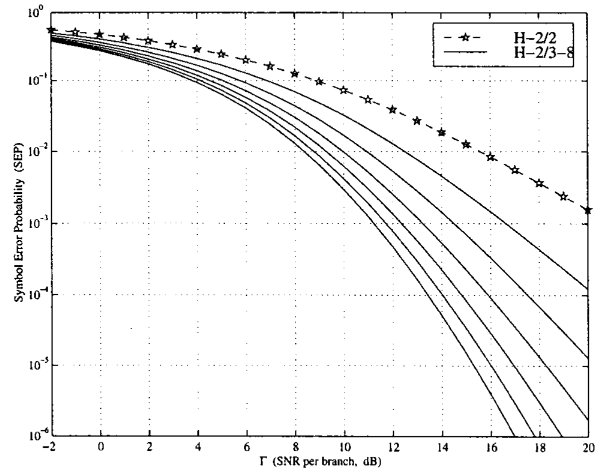

Fig. 3. The symbol error probability for coherent detection of 8-PSK with H-S/MRC as a function of average SNR per branch for various N with $L = 2$. The curves are parameterized by different H-L/N, starting from the highest curve representing H-2/2, and decrease monotonically to the lowest curve representing H-2/8.

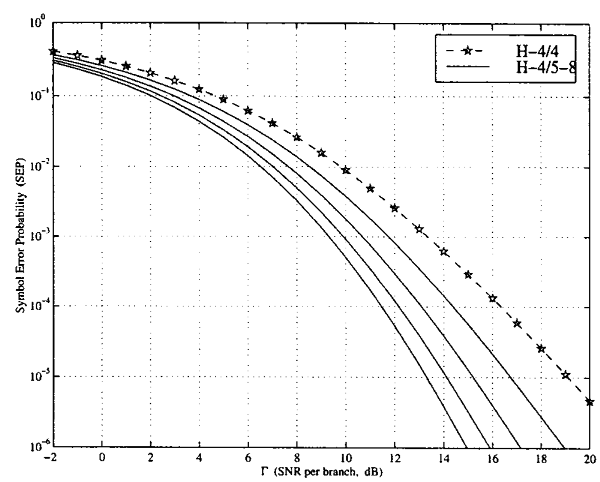

Fig. 4. The symbol error probability for coherent detection of 8-PSK with H-S/MRC as a function of average SNR per branch for various N with $L = 4$. The curves are parameterized by different H-L/N, starting from the highest curve representing H-4/4, and decrease monotonically to the lowest curve representing H-4/8.