Higher Order Statistics of the Output SNR of Hybrid Selection/Maximal-Ratio Combining

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Abstract—**In this paper, we use a virtual branch technique to derive higher order statistics of the output signal-to-noise ratio (SNR) of hybrid selection/maximal-ratio combining (H-S/MRC) in a multipath fading environment. In particular, the cumulants, central moments, skewness and kurtosis are derived. We consider the case of independent Rayleigh fading with equal receive SNR averaged over the fading on each diversity branch.**

I. Introduction

Hybrid selection/maximal-ratio combining (H-S/MRC) (see, for example, [1], [2]) is a diversity combining scheme where L (with the largest SNR at each instant) out of N diversity branches are selected and combined using maximal-ratio combining (MRC). This technique provides improved performance over L branch MRC when additional diversity is available, without requiring additional electronics and/or power consumption.

In a previous paper [2], we derived the mean and variance of the output SNR of H-S/MRC for any L and N under the assumption of independent Rayleigh fading with equal receive SNR averaged over the fading (on each diversity branch). This was performed using a "virtual branch" technique which simplified the derivations. The higher order statistics (HOS) (higher than second order) are also useful in signal processing algorithms for signal detection, classification and estimation as highlighted in [3], [4]. The use of HOS has seen increasing utility as manifested in [5].

In this paper, we extend the results of [2] to derive analytical expressions for the HOS of the output SNR of H-S/MRC. Specifically, we derive the cumulants, central moments, skewness and kurtosis of the output SNR. The proposed problem is made analytically tractable by transforming the ordered physical branches, which are necessarily dependent, into independent and identically distributed (i.i.d.) virtual branches, thereby permitting the derivation of the HOS expressions for arbitrary L and N.

In Section II, we present a system model of general diversity combining and derive the characteristic function (c.f.) of the output SNR using a virtual branch technique. The derivation of the HOS is given in Section III. Specifically, we derive the cumulants from the c.f., the central moments, and the skewness and kurtosis in Sections III-A, B, and C, respectively. In Section IV, the HOS for several diversity combining schemes including H-S/MRC, selection diversity (SD), and MRC are obtained from the general theory developed in Section III. Conclusions are given in Section V.

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II. Diversity Combining Analysis via the Virtual Branch Technique

A. System Model

Let γ_i denote the instantaneous SNR of the i^{th} diversity branch defined by $\gamma_i \triangleq \alpha_i^2 E_s/N_{0i}$, where $2E_s$ is the average symbol energy, and α_i is the instantaneous fading amplitude and $2N_{0i}$ is the two-sided noise power spectral density of the i^{th} branch. We model the α_i's as i.i.d. Rayleigh random variables (r.v.'s), and thus γ_i's are i.i.d. continuous r.v.'s with exponential p.d.f. and mean $\Gamma = \mathbb{E}\{\gamma_1\}$.

Let us first consider general diversity combining (GDC) with the instantaneous output SNR of the form

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} a_i \gamma_{(i)}\,, \tag{1}$$

where $a_i \in \{0,1\}$, $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)}$, and N is the number of available diversity branches. It will be apparent later that several diversity combining schemes, including H-S/MRC, turn out to be special cases of (1). Note that the possibility of at least two equal $\gamma_{(i)}$'s is excluded, since $\gamma_{(i)} \neq \gamma_{(j)}$ *almost surely* for continuous random variables.[1] Denoting $\boldsymbol{\gamma}_{(N)} \triangleq (\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(N)})$, the joint p.d.f. of $\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(N)}$ is [2]

$$f_{\boldsymbol{\gamma}_{(N)}}\left(\{\gamma_{(i)}\}_{i=1}^{N}\right) = \begin{cases} \frac{N!}{\Gamma^N} e^{-\frac{1}{\Gamma}\sum_{m=1}^{N}\gamma_{(m)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

It is important to note that the $\gamma_{(i)}$'s are *no* longer independent, even though the underlying γ_i's are independent.

[1]In our context, the notion of "almost surely" or "almost everywhere" can be stated mathematically as: if $\mathcal{N} = \{\gamma_{(i)} = \gamma_{(j)}\}$, then $\Pr\{\mathcal{N}\} = 0$ [6], [7].

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B. Characteristic Function for GDC

The characteristic function of the output SNR of GDC in a multipath-fading environment is given by

$$\psi_{\gamma_{GDC}}(j\nu) \triangleq \mathbb{E}_{\gamma_{GDC}}\{e^{+j\nu\gamma_{GDC}}\} = \int_0^{\infty} e^{+j\nu\gamma} f_{\gamma_{GDC}}(\gamma) d\gamma \,, \tag{3}$$

where $f_{\gamma_{GDC}}(\cdot)$ is the p.d.f. of the output SNR. Alternatively, the expectation in (3) can be obtained by substituting the expression for γ_{GDC} directly in terms of the physical branch variables given in (1) into (3) as

$$\begin{aligned}\psi_{\gamma_{GDC}}(j\nu) &= \mathbb{E}_{\{\gamma_{(i)}\}}\{e^{+j\nu\sum_{i=1}^{N} a_i\gamma_{(i)}}\} \\ &= \int_0^{\infty}\int_0^{\gamma_{(1)}}\cdots\int_0^{\gamma_{(N-1)}} e^{+j\nu\sum_{i=1}^{N} a_i\gamma_{(i)}} \\ &\times \quad f_{\gamma_{(N)}}\left(\{\gamma_{(i)}\}_{i=1}^{N}\right) d\gamma_{(N)}\cdots d\gamma_{(2)}d\gamma_{(1)} \,.\end{aligned} \tag{4}$$

Since the statistics of the ordered-branches are *no* longer independent, the evaluation of (4) involves nested N-fold integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR of the ordered physical diversity branches, $\gamma_{(i)}$, into a new set of *virtual branch* instantaneous SNR's, V_n's, using the following relation:

$$\gamma_{(i)} = \sum_{n=i}^{N} \frac{\Gamma}{n} V_n \,. \tag{5}$$

It can be verified that the instantaneous SNR's of the virtual branches are i.i.d. normalized exponential random variables with characteristic function given by

$$\psi_{V_n}(j\nu) \triangleq \mathbb{E}\left\{e^{+j\nu V_n}\right\} = \frac{1}{1-j\nu} \,. \tag{6}$$

The key advantage of the above transformation is that the instantaneous SNR of the output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{GDC} = \sum_{n=1}^{N} b_n V_n \,, \tag{7}$$

where the coefficients b_n are given by

$$b_n = \frac{\Gamma}{n}\sum_{i=1}^{n} a_i \,. \tag{8}$$

Using the *independent* virtual branches, the N-fold nested integrals of (4) reduce to

$$\psi_{\gamma_{GDC}}(j\nu) = \mathbb{E}_{\{V_n\}}\left\{e^{+j\nu\sum_{n=1}^{N} b_n V_n}\right\} = \prod_{n=1}^{N} \psi_{V_n}(j\nu b_n) \,. \tag{9}$$

Therefore

$$\psi_{\gamma_{GDC}}(j\nu) = \prod_{n=1}^{N} \frac{1}{1-j\nu b_n} \,. \tag{10}$$

The usefulness of the virtual branch technique is apparent by observing that the expectation operation in (4) no longer requires the N-fold nested integration.

III. Higher Order Statistics of GDC

A. Cumulants of GDC

The cumulants of GDC are given by

$$\kappa_{k,\mathrm{GDC}} = \frac{1}{j^k}\frac{d^k}{d\nu^k}\ln\psi_{\gamma_{GDC}}(j\nu)\Big|_{\nu=0} \,. \tag{11}$$

Note that the cumulants, except for κ_1, are invariant with respect to the translations of γ_{GDC}, hence they are also known as "semi-invariants." It can be shown that

$$\frac{d^k}{d\nu^k}\ln\psi_{\gamma_{GDC}}(j\nu) = (k-1)!\sum_{n=1}^{N}\frac{(jb_n)^k}{(1-j\nu b_n)^k} \,, \tag{12}$$

and hence

$$\kappa_{k,\mathrm{GDC}} = (k-1)!\sum_{n=1}^{N} b_n^k \,. \tag{13}$$

Note that the mean and variance of the output SNR of GDC can be obtained from (13) as

$$\Gamma_{\mathrm{GDC}} = \kappa_{1,\mathrm{GDC}} = \sum_{n=1}^{N} b_n \tag{14}$$

$$\sigma^2_{\mathrm{GDC}} = \kappa_{2,\mathrm{GDC}} = \sum_{n=1}^{N} b_n^2 \,, \tag{15}$$

which are in agreement with previous results.

B. Central Moments of GDC

The central moments of GDC are given by

$$\mu_{k,\mathrm{GDC}} \triangleq \mathbb{E}\left\{(\gamma_{\mathrm{GDC}} - \kappa_{1,\mathrm{GDC}})^k\right\} = \frac{1}{j^k}\frac{d^k}{d\nu^k}\Psi_{\gamma_{GDC}}(j\nu)\Big|_{\nu=0} = \frac{1}{j^k}\Psi^{(k)}_{\gamma_{GDC}}(0) \,, \tag{16}$$

where $\Psi_{\gamma_{GDC}}(j\nu)$ is the central c.f. of γ_{GDC} defined by

$$\Psi_{\gamma_{GDC}}(j\nu) \triangleq \mathbb{E}_{\gamma_{GDC}}\left\{e^{+j\nu(\gamma_{GDC}-\kappa_{1,GDC})}\right\} \,, \tag{17}$$

and $\Psi^{(k)}_{\gamma_{GDC}}(j\nu)$ denotes the $k^{\underline{\mathrm{th}}}$ derivative of $\Psi_{\gamma_{GDC}}(j\nu)$ with respect to the variable ν. Similar to the derivation of (10), $\Psi_{\gamma_{GDC}}(j\nu)$ can be derived using the virtual branch technique as

$$\Psi_{\gamma_{GDC}}(j\nu) = \prod_{n=1}^{N}\frac{e^{-j\nu b_n}}{(1-j\nu b_n)} \,. \tag{18}$$

To obtain the derivatives of $\Psi_{\gamma_{GDC}}(j\nu)$, we first consider the derivatives of

$$h(j\nu) = \ln\Psi_{\gamma_{GDC}}(j\nu) = \sum_{n=1}^{N}\left[-j\nu b_n - \ln(1-j\nu b_n)\right] \,. \tag{19}$$

It can be easily shown that

$$h^{(q)}(j\nu) = \begin{cases} \sum_{n=1}^{N} \left[-jb_n + \frac{jb_n}{(1-j\nu b_n)}\right], & q=1 \\ (q-1)! \sum_{n=1}^{N} \frac{(jb_n)^q}{(1-j\nu b_n)^q}, & q=2,3,\ldots \end{cases} \tag{20}$$

The $k^{\underline{\text{th}}}$ derivative of $\Psi_{\gamma_{\text{GDC}}}(j\nu)$ can be obtained using *Faa di Bruno's formula for the derivatives of a composite function* [8] as

$$\Psi^{(k)}_{\gamma_{\text{GDC}}}(j\nu) = \Psi_{\gamma_{\text{GDC}}}(j\nu) \times \sum_{\substack{(l_1,\ldots,l_k) \\ 0\le l_1,\ldots,l_k\le k \\ l_1+2l_2+\cdots+kl_k=k}} \frac{k!}{l_1!\cdots l_k!} \left[\frac{h^{(1)}(j\nu)}{1!}\right]^{l_1} \cdots \left[\frac{h^{(k)}(j\nu)}{k!}\right]^{l_k}. \tag{21}$$

From (20), we obtain

$$h^{(q)}(0) = \begin{cases} 0 & q=1 \\ (q-1)! \sum_{n=1}^{N} b_n^q, & q=2,3,\ldots \end{cases} \tag{22}$$

Since $h^{(1)}(0) = 0$, when evaluating $\Psi^{(k)}_{\gamma_{\text{GDC}}}(0)$ from (21), it is to be noted that all terms in the summation having $l_1 \ge 1$ will be zero. This implies that $\Psi^{(1)}_{\gamma_{\text{GDC}}}(0) = 0$ and we need to consider only terms with $l_1 = 0$ in evaluating $\Psi^{(k)}_{\gamma_{\text{GDC}}}(0)$ for $k > 1$. As a result, we obtain from (16), (21) and (22), and the fact that $\Psi_{\gamma_{\text{GDC}}}(0) = 1$, the expression

$$\mu_{k,\text{GDC}} = \begin{cases} 0 & k=1 \\ \sum_{\substack{(l_2,\ldots,l_k) \\ 0\le l_2,\ldots,l_k\le k \\ 2l_2+\cdots+kl_k=k}} \prod_{q=2}^{k} \frac{q}{l_q!} \left[\frac{1}{q}\sum_{n=1}^{N} b_n^q\right]^{l_q} & \text{otherwise}. \end{cases} \tag{23}$$

In the following, we list the first four central moments explicitly. For $k = 1$, (23) implies immediately that $\mu_{1,\text{GDC}} = 0$. For $k = 2$, it can be shown from (23) that $2l_2 + 3l_3 = 2$, which implies that $(l_2, l_3) = (1, 0)$, and hence

$$\mu_{2,\text{GDC}} = \sum_{n=1}^{N} b_n^2. \tag{24}$$

Note that $\mu_{1,\text{GDC}}$ and $\mu_{2,\text{GDC}}$ are in agreement with the results of [2]. Similarly, for $k = 3$, $2l_2 + 3l_3 = 3$ implies $(l_2, l_3) = (0, 1)$, and hence

$$\mu_{3,\text{GDC}} = 2\sum_{n=1}^{N} b_n^3, \tag{25}$$

and that for $k = 4$, $2l_2 + 3l_3 + 4l_4 = 4$ implies $(l_2, l_3, l_4) = (0, 0, 1), (2, 0, 0)$, and hence

$$\mu_{4,\text{GDC}} = 3\left(\sum_{n=1}^{N} b_n^2\right)^2 + 6\sum_{n=1}^{N} b_n^4. \tag{26}$$

C. Skewness and Kurtosis of GDC

The skewness of the output SNR of GDC can be obtained from either the cumulants derived in Section III-A or central moments derived in Section III-B as

$$\beta_{1,\text{GDC}} = \frac{\kappa_{3,\text{GDC}}}{\kappa_{2,\text{GDC}}^{3/2}} = \frac{\mu_{3,\text{GDC}}}{\mu_{2,\text{GDC}}^{3/2}} = \frac{2\sum_{n=1}^{N} b_n^3}{\left[\sum_{n=1}^{N} b_n^2\right]^{3/2}} \tag{27}$$

Similarly, the kurtosis of the output SNR of GDC can be obtained as

$$\beta_{2,\text{GDC}} = \frac{\kappa_{4,\text{GDC}}}{\kappa_{2,\text{GDC}}^2} + 3 = \frac{\mu_{4,\text{GDC}}}{\mu_{2,\text{GDC}}^2} = \frac{6\sum_{n=1}^{N} b_n^4}{\left[\sum_{n=1}^{N} b_n^2\right]^2} + 3, \tag{28}$$

IV. Application of General Theory

The results given in (13), (23), (27), and (28) are expressions for the cumulants, central moments, skewness and kurtosis, of the output SNR for N-branch GDC in Rayleigh-fading channels. The b_n's in (13), (23), (27), and (28) depend on the choice of a_i's via (8). These HOS expressions obtained in Section III are *general* in the sense that they apply to a variety of diversity combining systems that fit into the form of (1), including H-S/MRC, SD, and MRC.

A. H-S/MRC

In this section, the general theory derived in Section III is used to evaluate the HOS of H-S/MRC. The instantaneous output SNR of H-S/MRC is

$$\gamma_{\text{H-S/MRC}} = \sum_{i=1}^{L} \gamma_{(i)}, \tag{29}$$

where $1 \le L \le N$. Note that $\gamma_{\text{H-S/MRC}} = \gamma_{\text{GDC}}$ with

$$a_i = \begin{cases} 1, & i = 1,\ldots,L \\ 0, & \text{otherwise}. \end{cases} \tag{30}$$

In this case,

$$b_n = \begin{cases} \Gamma, & n \le L \\ \Gamma\frac{L}{n}, & \text{otherwise}. \end{cases} \tag{31}$$

Substituting (31) into (13), the cumulants of H-S/MRC can be easily obtained as

$$\kappa_{k,\text{H-S/MRC}} = (k-1)!\,\Gamma^k \left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^k\right]. \tag{32}$$

Similarly, the central moments of H-S/MRC can be obtained by substituting (31) into (23) as

$$\mu_{k,\text{H-S/MRC}} = \begin{cases} 0 & k=1 \\ \Gamma^k \sum\limits_{\substack{(l_2,\dots,l_k) \\ 0\le l_2,\dots,l_k\le k \\ 2l_2+\dots+kl_k=k}} \prod\limits_{q=2}^{k} \frac{q}{l_q!} \left\{ \frac{1}{q} \left[L + \sum\limits_{n=L+1}^{N} \left(\frac{L}{n}\right)^q \right] \right\}^{l_q} & \text{otherwise}. \end{cases} \tag{33}$$

The first central moment is equal to zero as required. The second, third, and fourth central moments are obtained by substituting (31) into (24), (25), and (26) as

$$\mu_{2,\text{H-S/MRC}} = \Gamma^2 \left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^2 \right], \tag{34}$$

$$\mu_{3,\text{H-S/MRC}} = 2\Gamma^3 \left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^3 \right], \tag{35}$$

and

$$\mu_{4,\text{H-S/MRC}} = 3\Gamma^4 \left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^2 \right]^2 + 6\Gamma^4 \left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^4 \right]. \tag{36}$$

The skewness and kurtosis of H-S/MRC can be obtained by substituting (31) into (27) and (28) respectively as

$$\beta_{1,\text{H-S/MRC}} = \frac{2\left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^3 \right]}{\left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^2 \right]^{3/2}}, \tag{37}$$

and

$$\beta_{2,\text{H-S/MRC}} = \frac{6\left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^4 \right]}{\left[L + \sum_{n=L+1}^{N} \left(\frac{L}{n}\right)^2 \right]^{2}} + 3. \tag{38}$$

B. SD

SD is the simplest form of diversity combining whereby the received signal from *one* of N diversity branches is selected [9]. The output SNR of SD is

$$\gamma_{\text{SD}} \triangleq \max_i \{\gamma_i\} = \gamma_{(1)}. \tag{39}$$

Note that SD is a special case of H-S/MRC with $L = 1$. Substituting $L = 1$ into (32) and (33), the cumulants and central moments of SD can be easily obtained as

$$\kappa_{k,\text{SD}} = (k-1)!\,\Gamma^k \sum_{n=1}^{N} \frac{1}{n^k}, \tag{40}$$

and

$$\mu_{k,\text{SD}} = \begin{cases} 0 & k=1 \\ \Gamma^k \sum\limits_{\substack{(l_2,\dots,l_k) \\ 0\le l_2,\dots,l_k\le k \\ 2l_2+\dots+kl_k=k}} \prod\limits_{q=2}^{k} \frac{q}{l_q!} \left[\frac{1}{q} \sum\limits_{n=1}^{N} \frac{1}{n^q} \right]^{l_q} & \text{otherwise}, \end{cases} \tag{41}$$

respectively. From (41) the second, third and fourth central moments of SD are

$$\mu_{2,\text{SD}} = \Gamma^2 \sum_{n=1}^{N} \frac{1}{n^2}, \tag{42}$$

$$\mu_{3,\text{SD}} = 2\Gamma^3 \sum_{n=1}^{N} \frac{1}{n^3}, \tag{43}$$

and

$$\mu_{4,\text{SD}} = 3\Gamma^4 \left[\sum_{n=1}^{N} \frac{1}{n^2} \right]^2 + 6\Gamma^4 \sum_{n=1}^{N} \frac{1}{n^4}. \tag{44}$$

Similarly, the skewness and kurtosis of SD are

$$\beta_{1,\text{SD}} = \frac{2\left[\sum_{n=1}^{N} \frac{1}{n^3} \right]}{\left[\sum_{n=1}^{N} \frac{1}{n^2} \right]^{3/2}} \tag{45}$$

and

$$\beta_{2,\text{SD}} = \frac{6\left[\sum_{n=1}^{N} \frac{1}{n^4} \right]}{\left[\sum_{n=1}^{N} \frac{1}{n^2} \right]^{2}} + 3, \tag{46}$$

respectively.

Note that for large N,

$$\sum_{n=1}^{N} \frac{1}{n^p} \approx \zeta(p), \quad p > 1, \tag{47}$$

where $\zeta(.)$ denotes the *Weierstrass's zeta function* (see [10], 0.233, page 9). In particular,

$$\zeta(2) = \frac{\pi^2}{6} = 1.6449, \quad \zeta(3) = 1.2021, \quad \zeta(4) = \frac{\pi^4}{90} = 1.0823. \tag{48}$$

Therefore when $N \gg 1$, we get the following asymptotic results for the cumulants and central moments by applying (47) in (40), (41), (42), (43), (44) respectively:

$$\kappa_{k,\text{SD}} \approx (k-1)!\,\Gamma^k\,\zeta(k), \quad k > 1, \tag{49}$$

and

$$\mu_{k,\mathrm{SD}} \approx \begin{cases} 0 & k=1 \\ \Gamma^k \sum_{\substack{(l_2,\ldots,l_k) \\ 0\le l_2,\ldots,l_k\le k \\ 2l_2+\cdots+kl_k=k}} \prod_{q=2}^{k} \frac{q}{l_q!}\left[\frac{\zeta(q)}{q}\right]^{l_q} & \text{otherwise}; \end{cases} \tag{50}$$

$$\mu_{2,\mathrm{SD}} \approx 1.6449\ \Gamma^2\,, \tag{51}$$

$$\mu_{3,\mathrm{SD}} \approx 2.4042\ \Gamma^3\,, \tag{52}$$

and

$$\mu_{4,\mathrm{SD}} \approx 14.6114\ \Gamma^4\,. \tag{53}$$

Similarly, from (45) and (46), the asymptotic values of the skewness and kurtosis of SD for large N are given by

$$\beta_{1,\mathrm{SD}} \approx 1.1396 \tag{54}$$

and

$$\beta_{2,\mathrm{SD}} \approx 5.4 \tag{55}$$

respectively.

C. MRC

In MRC, the received signals from *all* diversity branches are weighted and combined to maximize the SNR at the output [9]. The output SNR of MRC is

$$\gamma_{\mathrm{MRC}} \triangleq \sum_{i=1}^{N}\gamma_i = \sum_{i=1}^{N}\gamma_{(i)}\,. \tag{56}$$

Since MRC is a special case of H-S/MRC with $L = N$, the cumulants and central moments of MRC are obtained respectively by setting $L = N$ in (32) and (33) as

$$\kappa_{k,\mathrm{MRC}} = (k-1)!\left[N\Gamma^k\right]\,, \tag{57}$$

and

$$\mu_{k,\mathrm{MRC}} = \begin{cases} 0 & k=1 \\ \Gamma^k \sum_{\substack{(l_2,\ldots,l_k) \\ 0\le l_2,\ldots,l_k\le k \\ 2l_2+\cdots+kl_k=k}} \prod_{q=2}^{k} \frac{q}{l_q!}\left[\frac{N}{q}\right]^{l_q} & \text{otherwise}. \end{cases} \tag{58}$$

The second, third and fourth central moments of MRC are

$$\mu_{2,\mathrm{MRC}} = N\Gamma^2\,, \tag{59}$$

$$\mu_{3,\mathrm{MRC}} = 2N\Gamma^3\,, \tag{60}$$

and

$$\mu_{4,\mathrm{MRC}} = 3N(N+2)\Gamma^4\,. \tag{61}$$

The skewness and kurtosis of MRC become

$$\beta_{1,\mathrm{MRC}} = \frac{2}{\sqrt{N}}\,, \tag{62}$$

and

$$\beta_{2,\mathrm{MRC}} = \frac{6}{N} + 3\,. \tag{63}$$

For large N, we have

$$\beta_{1,\mathrm{MRC}} \approx 0\,, \tag{64}$$

and

$$\beta_{2,\mathrm{MRC}} \approx 3\,. \tag{65}$$

V. Conclusions

We derived exact expressions for the higher order statistics, in particular, cumulants, central moments, skewness and kurtosis, of the output SNR of GDC in Rayleigh-fading. We analyzed this system using a "virtual branch" technique which resulted in a simple derivation of the HOS expressions for *arbitrary* L and N. The HOS of H-S/MRC, SD, and MRC were obtained as special cases of our results.

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