A Simple and Asymptotically Tight Upper Bound on the Symbol Error Probability of Adaptive Antennas with Optimum Combining

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Abstract— **We derive a simple closed-form upper bound on the symbol error probability for coherent detection of M-ary PSK using an array of antennas with optimum combining. We assume multiple equal-power cochannel interferers and thermal noise in a Rayleigh fading environment. The new bound applies for an arbitrary number of antenna elements as well as arbitrary number of interferers, and it is proved to be asymptotically tight. Based on the simplicity of the bound, the signal-to-noise ratio penalty due to co-channel interference is evaluated. Comparisons with simulation is also provided, showing that our bound is useful in a large number of practical interesting cases.**

Keywords— **Antenna diversity, adaptive arrays, optimum combining, cochannel interference, Wishart matrices, eigenvalues distribution.**

I. Introduction

Adaptive antennas can significantly improve the performance of wireless communication systems by suitably combining the received signals to reduce fading effects and suppress interference. In particular, with optimum combining the received signals are weighted and combined to maximize the output signal-to-interference-plus-noise power ratio (SINR). This technique provides substantial improvement in performance over maximal ratio combining (MRC), where the received signals are combined to maximize the desired signal power only, when interference is present.

Exact expressions for the bit error probability (BEP) of binary phase-shift keying (PSK) have been derived for the single interferer case with Rayleigh fading of the desired signal in [1] and with Rayleigh fading of the desired signal and interferer in [2, 3].

With multiple interferers of arbitrary power, Monte Carlo simulation has been used to determine the BEP in [1]. To avoid Monte Carlo simulation, the exact BEP expression was derived in [4] in the absence of thermal noise and for equal-power interferers. Various approximations for the BEP have been presented in [5, 6] for binary modulation in the presence of thermal noise.

The only upper bounds on the BEP of optimum combining including thermal noise and an arbitrary number of interferers were derived in [7], given the average powers of the interferers. However, these bounds are generally not tight and the use of them requires integer coefficients. The required coefficients, dependent on the number of interferers, were tabulated in [7] for the number of interferers up to 7. To use the bounds for larger number of interferers necessitates the generation of these integer coefficients whose computation complexity grows exponentially with the number of antenna elements.

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An interesting subclass of interferers is the equal-power interferers case, which generally arises in multiple-input-multiple-output (MIMO) systems [8]; for example, when the co-channel interferer is a MIMO user, multiple equal-power interferers are present.

In this paper, starting from an approach similar to that used in [7], we apply some results on characteristic polynomial of a complex Wishart matrices to derive new simple upper bounds of the symbol error probability (SEP) for coherent detection of M-ary PSK using optimum combining in the presence of multiple equal-power interferers, as well as thermal noise, in a slow Rayleigh fading environment.

In Section II we describe the system model; performance and upper bounds are derived in Section III, and in Section IV we compare our analytical bounds with simulations.

II. System Model

The received signal at the $N_{\rm A}$-element array output consists of the desired signal, $N_{\rm I}$ interfering signals, and thermal noise. After matched filtering and sampling at the symbol rate, the array output vector at time k can be written as:

$$\mathbf{z}(k) = \sqrt{P_{\rm D}}\boldsymbol{c}_{\rm D} b_0(k) + \sqrt{P_{\rm I}} \sum_{j=1}^{N_{\rm I}} \boldsymbol{c}_{{\rm I},j} b_j(k) + \mathbf{n}(k) \, , \qquad (1)$$

where $P_{\rm D}$ and $P_{\rm I}$ are the mean (over fading) power of the desired and interfering signal, respectively; $\boldsymbol{c}_{\rm D}$ and $\boldsymbol{c}_{{\rm I},j}$ are the desired and $j^{\underline{\rm th}}$ interfering signal propagation vectors, respectively; $b_0(k)$ and $b_j(k)$ (both with unit variance) are the desired and interfering data samples, respectively; and $\mathbf{n}(k)$ represents the additive noise.[1]

The vectors $\boldsymbol{c}_{\rm D}$ and $\boldsymbol{c}_{{\rm I},j}$ are multivariate complex-valued Gaussian vectors having $\mathbb{E}\{\boldsymbol{c}_{\rm D}\} = \mathbb{E}\{\boldsymbol{c}_{{\rm I},j}\} = \mathbf{0}$ and $\mathbb{E}\left\{\boldsymbol{c}_{\rm D}\boldsymbol{c}_{\rm D}^{\dagger}\right\} = \mathbb{E}\left\{\boldsymbol{c}_{{\rm I},j}\boldsymbol{c}_{{\rm I},j}^{\dagger}\right\} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The $j^{\underline{\rm th}}$ interfering data samples, $b_j(k)$ $j = 1, \ldots, N_{\rm I}$, can be modeled as zero-mean, unitary variance Gaussian random variables. The additive noise is modeled as a white Gaussian random vector with independent and identically distributed (i.i.d.) elements with $\mathbb{E}\{\mathbf{n}(k)\} = \mathbf{0}$ and $\mathbb{E}\left\{\mathbf{n}(k)\mathbf{n}^{\dagger}(k)\right\} = \sigma^2\mathbf{I}$, where σ^2 is the thermal noise power per antenna element.

[1] Throughout the paper $(\cdot)^T$ denotes the transposition operator, and $(\cdot)^{\dagger}$ stands for conjugation and transposition.

We also define the signal-to-noise ratio (SNR) as $P_{\rm D}/\sigma^2$ and the signal power to total interfering power ratio (SIR) as $P_{\rm D}/(N_{\rm I} \cdot P_{\rm I})$.

The SINR at the output of the $N_{\rm A}$-element array with optimum combining can be expressed [1] as

$$\gamma = P_{\rm D} \boldsymbol{c}_{\rm D}^{\dagger} \mathbf{R}^{-1} \boldsymbol{c}_{\rm D} \,, \tag{2}$$

where the short-term covariance matrix $\mathbf{R}$, conditioned to all interference propagation vectors, is given by

$$\mathbf{R} = P_{\rm I} \sum_{j=1}^{N_{\rm I}} \boldsymbol{c}_{{\rm I},j} \boldsymbol{c}_{{\rm I},j}^{\dagger} + \sigma^2 \mathbf{I}. \tag{3}$$

It is important to remark that $\mathbf{R}$, and consequently also the SINR γ, varies at the fading rate, which is assumed to be much slower than the symbol rate.

The matrix $\mathbf{R}^{-1}$ can be written as $\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{\dagger}$ where $\mathbf{U}$ is a unitary matrix and $\mathbf{\Lambda}$ is a diagonal matrix whose elements on the principal diagonal are the eigenvalues of $\mathbf{R}$, denoted by $(\lambda_1, \ldots, \lambda_{N_{\rm A}})$. The vector $\mathbf{u} = \mathbf{U}^{\dagger}\boldsymbol{c}_{\rm D} = [u_1, ..., u_{N_{\rm A}}]^T$ has the same distribution as $\boldsymbol{c}_{\rm D}$, since $\mathbf{U}$ represents a unitary transformation. The SINR given in (2) can be rewritten as:

$$\gamma = P_{\rm D} \boldsymbol{c}_{\rm D}^{\dagger} \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{\dagger} \boldsymbol{c}_{\rm D} = P_{\rm D} \sum_{i=1}^{N_{\rm A}} \frac{|u_i|^2}{\lambda_i} \,. \tag{4}$$

Since $\mathbf{R}$ is a random matrix, its eigenvalues are random variables.

By introducing the notation $N_{\rm min} = \min\{N_{\rm I}, N_{\rm A}\}$ and $N_{\rm max} = \max\{N_{\rm I}, N_{\rm A}\}$, the eigenvalues of $\mathbf{R}$ can be shown that

$$\lambda_i = \begin{cases} P_{\rm I}\tilde{\lambda}_i + \sigma^2, & i = 1, \ldots, N_{\rm min} \,. \\ \sigma^2, & i = N_{\rm min}+1, \ldots, N_{\rm A} \,. \end{cases} \tag{5}$$

where $\tilde{\lambda}_i$ are eigenvalues of complex Wishart matrix, denoted by $\tilde{\mathbf{W}}(N_{\rm min}, N_{\rm max})$ [14]. Hence, the first $N_{\rm min}$ eigenvalues λ_i with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_{\rm min}}$ can be thought as those of the $(N_{\rm min} \times N_{\rm min})$ matrix

$$P_{\rm I}\tilde{\mathbf{W}}(N_{\rm min}, N_{\rm max}) + \sigma^2 \mathbf{I} \,. \tag{6}$$

III. Performace Analysis of Optimum Combining

A. Exact Symbol Error Probability

The SEP for optimum combining in the presence of multiple cochannel interferers and thermal noise in a fading environment is thus obtained by averaging the conditional SEP over the (desired and interfering signal) channel ensemble as $P_e = \mathbb{E}_{\gamma}\{\Pr\{e \mid \gamma\}\}$, where $\Pr\{e \mid \gamma\}$ is the SEP conditioned on the random variable γ. This can be accomplished by using the chain rule of conditional expectation as

$$P_e = \mathbb{E}_{\boldsymbol{\lambda}}\left\{ \underbrace{\mathbb{E}_{\mathbf{u}}\{\Pr\{e \mid \gamma\}\}}_{P_{e|\boldsymbol{\lambda}}} \right\} , \tag{7}$$

where we first perform $\mathbb{E}_{\mathbf{u}}\{\cdot\}$ (i.e., average over the channel ensemble of the desired signal) to obtain the conditional SEP, conditioned on the random vector $\boldsymbol{\lambda}$, denoted by $P_{e|\boldsymbol{\lambda}}$. We then perform $\mathbb{E}_{\boldsymbol{\lambda}}\{\cdot\}$ to average out the channel ensemble of the interfering signals.

Under the assumption of Gaussian interference and noise, $\Pr\{e \mid \gamma\}$ for coherent detection of M-ary PSK is given by [9]

$$\Pr\{e \mid \gamma\} = \frac{1}{\pi} \int_0^{\Theta} \exp\left(-\frac{c_{\rm MPSK}}{\sin^2\theta}\gamma\right) d\theta \,, \tag{8}$$

where $c_{\rm MPSK} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. Using (8) together with the fact that $\mathbf{u}$ is Gaussian with i.i.d. elements, the conditional SEP $P_{e|\boldsymbol{\lambda}}$, conditioned on $\boldsymbol{\lambda}$, in the general case of $N_{\rm A}$ antennas and $N_{\rm I}$ interferers becomes:

$$\begin{aligned} P_{e|\boldsymbol{\lambda}} &= \frac{1}{\pi}\int_0^{\Theta} \mathbb{E}_{\mathbf{u}}\left\{\exp\left(-\frac{c_{\rm MPSK}}{\sin^2\theta}P_{\rm D}\sum_{i=1}^{N_{\rm A}}\frac{|u_i|^2}{\lambda_i}\right)\right\} d\theta \\ &= \frac{1}{\pi}\int_0^{\Theta} A(\theta)\prod_{i=1}^{N_{\rm min}}\left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\frac{P_{\rm D}}{\lambda_i}}\right] d\theta \,, \end{aligned} \tag{9}$$

where

$$A(\theta) \triangleq \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\frac{P_{\rm D}}{\sigma^2}}\right]^{N_{\rm A}-N_{\rm min}} . \tag{10}$$

B. Upper Bound

In this section we derive a new upper bound for the SEP based on the expected characteristic polynomial of a complex Wishart matrix.

Theorem 1: The SEP with optimum combining is upper bounded as follows:

$$P_e \leq N_{\rm min}! \, L_{N_{\rm min}}^{N_{\rm max}-N_{\rm min}}(-\Gamma_{\rm I}^{-1}) \cdot \Gamma_{\rm I}^{N_{\rm min}} P_{e,\rm MRC}\left(N_{\rm A}, SNR\right) , \tag{11}$$

where $\Gamma_{\rm I} \triangleq \frac{P_{\rm I}}{\sigma^2} = \frac{SNR}{N_{\rm I}\, SIR}$, $SNR \triangleq \frac{P_{\rm D}}{\sigma^2}$,

$$P_{e,\rm MRC}\left(N_{\rm A}, SNR\right) \triangleq \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\, SNR}\right]^{N_{\rm A}} d\theta \,, \tag{12}$$

and $L_n^m(x) = \sum_{k=0}^{n}\binom{n+m}{n-k}\frac{(-x)^k}{k!}$ are the generalized Laguerre polynomials [10, p. 1061, eq. (8.970)].

Proof:

Let us consider the conditional SEP of (9), and rewrite

$$\prod_{i=1}^{N_{\rm min}}\left[\frac{\sin^2\theta}{\sin^2\theta + \frac{c_{\rm MPSK}P_{\rm D}}{\lambda_i}}\right] = \frac{\prod_{i=1}^{N_{\rm min}}\frac{\lambda_i \sin^2\theta}{c_{\rm MPSK}P_{\rm D}}}{\prod_{i=1}^{N_{\rm min}}\left[1 + \frac{\lambda_i\sin^2\theta}{c_{\rm MPSK}P_{\rm D}}\right]} . \tag{13}$$

Then, by remembering that $\lambda_i = P_{\rm I}\tilde{\lambda}_i + \sigma^2$, $P_{\rm I} > 0$, and $\tilde{\lambda}_i$ are real and non-negative, the following inequality holds

$$\prod_{i=1}^{N_{\rm min}}\left[1 + \frac{\lambda_i\sin^2\theta}{c_{\rm MPSK}P_{\rm D}}\right] \geq \prod_{i=1}^{N_{\rm min}}\left[1 + \frac{\sigma^2\sin^2\theta}{c_{\rm MPSK}P_{\rm D}}\right] . \tag{14}$$

Therefore, by using (13) and (14), equation (9) can be upper bounded as follows

$$P_{e|\boldsymbol{\lambda}} \leq \frac{1}{\sigma^{2N_{\min}}} \left(\prod_{i=1}^{N_{\min}} \lambda_i \right) \frac{1}{\pi} \int_0^{\Theta} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{P_{\text{D}}}{\sigma^2}} \right]^{N_{\text{A}}} d\theta. \quad (15)$$

Now, in order to apply (7), we need the expectation

$$\mathbb{E}_{\boldsymbol{\lambda}}\left\{ \prod_{i=1}^{N_{\min}} \lambda_i \right\} = \mathbb{E}_{\boldsymbol{\lambda}}\left\{ det \left[P_{\text{I}} \tilde{\mathbf{W}}(N_{\min}, N_{\max}) + \sigma^2 \mathbf{I} \right] \right\},$$

where the last equality is due to (6). Starting from [11, p. 86] it is possible to show, in general, that the expectation of the characteristic polynomial of a complex Wishart matrix $\tilde{\mathbf{W}}(m,p)$ can be written as

$$\mathbb{E}\left\{ det \left[-x\mathbf{I} + \tilde{\mathbf{W}}(m,p) \right] \right\} = L_m^{p-m}(x) m! \quad (16)$$

and therefore

$$\mathbb{E}_{\boldsymbol{\lambda}}\left\{ det \left[P_{\text{I}} \tilde{\mathbf{W}}(N_{\min}, N_{\max}) + \sigma^2 \mathbf{I} \right] \right\} = P_{\text{I}}^{N_{\min}} N_{\min}! \, L_{N_{\min}}^{N_{\max}-N_{\min}} (-\Gamma_{\text{I}}^{-1}) \quad (17)$$

Finally, by using (15) and (17) we obtain (11).

■

C. Observations

Comparing (12) with [12, eq. (39)], we see that (12) is the exact expression of the SEP for coherent detection of M-ary PSK using N_{A}-branch MRC. Note that integral in (12) can evaluated in closed form [13]; however, we prefer to leave it since it is in a compact form and displays the dependence of SEP on SNR. Numerical evaluation of (12) is straightforward since the integrand of (12) is a simple expression involving trigonometric functions and the integration limits are finite.

It can be observed that (11) is asymptotically tight for $P_{\text{I}} \to 0$: in fact, in this case as $\lambda_i \to \sigma^2$ the inequality in (14) becomes equality and our bound gives the exact solution. It can be also verified that $N_{\min}! \, L_{N_{\min}}^{N_{\max}-N_{\min}} (-\Gamma_{\text{I}}^{-1}) \cdot \Gamma_{\text{I}}^{N_{\min}}$ approaches 1 as $\Gamma_{\text{I}} \to 0$ (or equivalently as $P_{\text{I}} \to 0$) and hence, as expected,the performance of OC in the absence of interference reduces to the performance of MRC.

In general, by expanding the Laguerre polynomial it can be seen that

$$N_{\min}! \, L_{N_{\min}}^{N_{\max}-N_{\min}} (-\Gamma_{\text{I}}^{-1}) \cdot \Gamma_{\text{I}}^{N_{\min}} = = 1 + a_1 \Gamma_{\text{I}} + \cdots + a_{N_{\min}} \Gamma_{\text{I}}^{N_{\min}} \quad (18)$$

is a monic polynomial with non-negative coefficients in Γ_{I} of degree $N_{\min}$: it is therefore greater than or equal to 1, and it represents the upper bound of the increase in SEP due to the presence of interfering signals.

IV. Numerical Results

To assess the validity of the proposed bound, in Fig. 1 we compare the SEP vs. SNR for BPSK with optimum combining as from (11) and from [7, eq. (13)].

The curves have been obtained with $N_{\text{A}} = 4$, SIR = 10 dB and N_{I}=1 and 4 interferers. It can be observed that the new simple bound is several dB's tighter than the previously known bound: e.g. the difference is more than 4 dB at SEP $= 10^{-3}$. Moreover, some comparisons with semi-analytical results are shown; these results are obtained by generating the random propagation vectors, computing the SINR by (2) and then the error probability by (8). Performance of adaptive antenna in the presence of interference is degraded when compared to the that in the absence of interference: the required SNR to achieve a specified SEP in the presence of interference increases when compared to the case without interference. We define the penalty ΔSNR as the difference in the two cases.

The upper bound on the SNR penalty is reported in Fig. 2 for an 8-PSK modulation and target SEP of 10^{-3}, with a SIR ranging from 10 to 20 dB's. In the figure the importance of using smart antennas can be appreciated, and the role played by the number of receiving antenna elements is clearly shown.

V. Conclusions

In this paper we derived, in closed-form, a simple asymptotically tight bound on the symbol error probability for optimum combining of signals in the presence of multiple interferers and thermal noise. Both cases $N_{\text{A}} \leq N_{\text{I}}$ and $N_{\text{A}} > N_{\text{I}}$ were investigated and compared with previous bounds. Moreover, to validate the tightness of the proposed model, a semi-analytical tool, obtained by generating the random propagation vectors, has been used. The penalty in SNR due to the presence of inter-

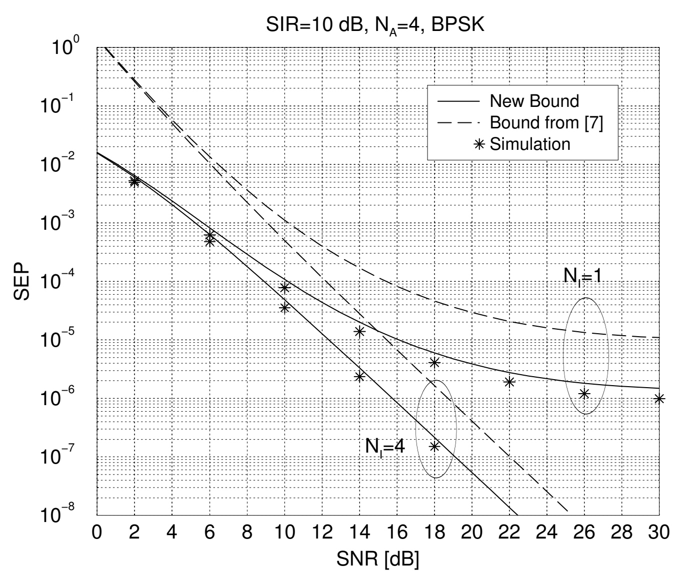

Fig. 1. Bounds comparison, BPSK, SIR=10 dB, $N_A = 4$ antennas. Semi-analytical results for N_I=1 and 4 are also shown.

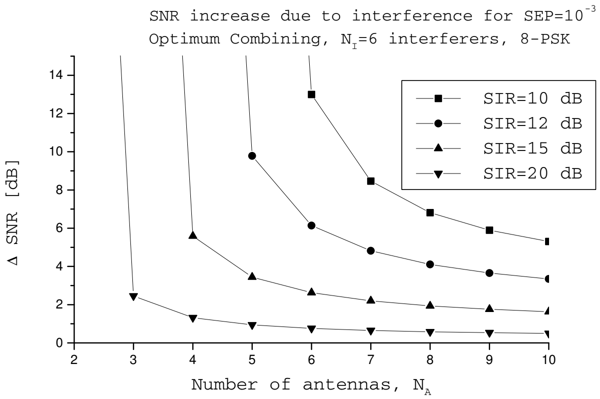

Fig. 2. Penalty on the SNR due to interference, target $SEP = 10^{-3}$, 8-PSK, N_I=6 interferers.

ference has been also introduced.

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