Bit Error Outage for Diversity Reception in Shadowing Environment

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***Abstract*—This letter addresses the problem of evaluating the bit error outage (BEO), i.e., the outage probability defined in terms of bit error probability, in a Rayleigh fading and shadowing environment. We consider coherent detection of binary phase-shift keying with maximal ratio combining (MRC). As an example application, the BEO in a log-normal shadowing environment is analyzed and the improvement in terms of BEO due to MRC is quantified in different shadowing environments.**

***Index Terms*—Bit error outage (BEO), diversity reception, fading channel.**

I. Introduction

PERFORMANCE for diversity systems in terms of symbol and bit error probability (BEP) (both averaged over the multipath or multichannel fading) has been extensively studied in the literature, with direct applications to antenna diversity and Rake reception [1]–[9]. However, the explicit expression for the inverse BEP (i.e., signal-to-noise ratio (SNR) as a function of BEP), required for many important problems related to digital mobile radio, is not known in general even in cases where closed-form BEP expressions are available. One noticeable example is provided by the bit error outage (BEO), i.e., the probability that the BEP exceeds a maximum tolerable level. This definition of outage probability is appropriate for digital communication systems where fast fading is superimposed on a slow fading. Derivation of such a BEO requires an inverse BEP expression that is not straightforward to obtain as it involves a numerical roots evaluation.

In this paper, we analyze the BEO for multichannel reception with maximum ratio combining (MRC) in a Rayleigh fading and shadowing environment. We first derive upper and lower bounds on the inverse BEP which are not only simple and explicit function of the target BEP, but also sufficiently tight for *all* values of SNR. We then replace the implicit inverse BEP expression with the above mentioned bounds which alleviates the analytical difficulty in BEO derivation and avoids numerical root evaluations.

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II. Bounds on the BEP and its Inverse

The instantaneous BEP expression for binary phase-shift keying (BPSK) in an additive white Gaussian noise channel is given by [3] $P_b(\gamma_{\text{tot}}) = Q(\sqrt{2\gamma_{\text{tot}}})$.[1] The BEP for coherent detection of BPSK with MRC in a multipath environment is obtained by averaging the instantaneous BEP over the fast fading process.[2] This can be obtained by using the alternative expression for the instantaneous BEP together with the characteristic function method (see, for example, [4], [9]) as

$$P_b(\overline{\gamma}) = \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2\theta}{\sin^2\theta+\overline{\gamma}}\right)^N d\theta \qquad (1)$$

where $\overline{\gamma}$ is the branch-SNR and N is the diversity order.[3] The form of (1) allow us to derive invertible BEP bounds.

By adopting the Chernoff–Rubin[4] bound $Q(x) \leq e^{-x^2/2}$ [3], [11] and averaging over instantaneous SNR distribution we obtain the following upper bound on the BEP:

$$P_b(\overline{\gamma}) \leq \left(\frac{1}{1+\overline{\gamma}}\right)^N \leq \frac{1}{\overline{\gamma}^N}. \qquad (2)$$

This expression is widely used in the coding literature (see, for example, [12] and [13, p. 718]) to bound the pairwise error probability over Rayleigh fading channels. In that context, the exponent N represents the Hamming distance of the coded sequences and plays the same role as the diversity order in multiple antenna reception. The result (2) derived by the Chernoff-Rubin inequality can be improved by a factor of 1/2 using the bound $Q(x) \leq (1/2)e^{-x^2/2}$ [14, p. 123].[5] In this way we have the improved bound for the BEP of BPSK over Rayleigh fading

$$P_b(\overline{\gamma}) \leq \frac{1}{2}\left(\frac{1}{1+\overline{\gamma}}\right)^N \leq \frac{1}{2\overline{\gamma}^N}. \qquad (3)$$

[1]The Gaussian Q-function $Q(x) \triangleq (1/\sqrt{2\pi})\int_x^{+\infty}\exp(-y^2/2)\,dy$ is related to the complementary error function by $Q(x) = (1/2)\text{erfc}\,(x/\sqrt{2})$.

[2]Unless otherwise stated, the terms BEP and SNR will be used in the following to denote the mean BEP and the mean SNR (averaged over the fast fading).

[3]We will use the terms "path" and "branch" interchangeably since our analysis applies to spatial diversity (i.e., antenna diversity) as well as time diversity (i.e., Rake reception).

[4]We will refer to as Chernoff–Rubin bound to reflect also the contribution of Herman Rubin, although it is usually referred to as Chernoff bound [10].

[5]Sometimes this is improperly referred to as the Chernoff bound.

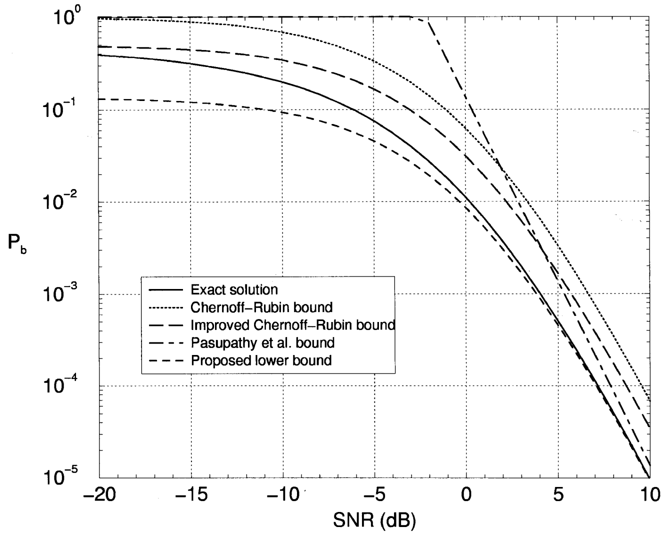

Fig. 1. $P_b(\overline{\gamma})$, lower and upper bounds for BPSK with $N = 4$.

Note that (2) and (3) are not asymptotically tight. The best known asymptotic behavior of (1) for BPSK is derived in [3] and [15] as

$$P_b(\overline{\gamma}) \approx \frac{S(N)}{\overline{\gamma}^N}, \tag{4}$$

where $S(N) = (1/2^{2N+1})\binom{2N}{N}$. In [15] it was observed that (4) is also an upper bound. Note that it gives an asymptotically tighter result than the bounds (2) and (3). Next we give a new concise proof of this fact using (1). By noting that $0 \leq \sin^2\theta \leq 1$, and replacing $\sin^2\theta$ with its minimum value, 0, in the denominator of the integrand function of (1), we immediately obtain $P_b(\overline{\gamma}) \leq (S(N)/\overline{\gamma}^N)$. This suggests that we can also obtain a lower bound for $P_b(\overline{\gamma})$ by replacing $\sin^2(\theta)$ with its maximum value, 1, in the denominator of the integrand function of (1), giving

$$P_{b,L}(\overline{\gamma}) \triangleq \frac{S(N)}{(1+\overline{\gamma})^N} \leq P_b(\overline{\gamma}). \tag{5}$$

Fig. 1 provides a comparison among previously discussed upper bounds together with the new lower bound for BPSK with $N = 4$. In general, (3) or (4) can be closer to the exact solution depending on N and on the P_b of interest (e.g., 10^{-2} or 10^{-3}). Hence, (5) provides us a lower bound, $P_{b,L}(\overline{\gamma})$; whereas the minimum between (3) and (4) provides us an upper bound, $P_{b,U}(\overline{\gamma})$. Note that both lower and upper bounds on $P_b(\overline{\gamma})$ are invertible. From $P_{b,L}(\overline{\gamma})$ and $P_{b,U}(\overline{\gamma})$ the required SNR, $\overline{\gamma}^\star = P_b^{-1}(P_b^\star)$, to achieve a target BEP with $P_b^\star \in [0, S(N)]$ can be lower and upper bounded by

$$\overline{\gamma}_L^\star(P_b^\star) \leq \overline{\gamma}^\star(P_b^\star) \leq \overline{\gamma}_U^\star(P_b^\star) \tag{6}$$

where

$$\overline{\gamma}_L^\star(P_b^\star) = \left[\frac{S(N)}{P_b^\star}\right]^{1/N} - 1 \tag{7a}$$

$$\overline{\gamma}_U^\star(P_b^\star) = \min\left\{\left(\frac{1}{2P_b^\star}\right)^{1/N} - 1, \left[\frac{S(N)}{P_b^\star}\right]^{1/N}\right\}. \tag{7b}$$

In fact, it is clear from the definitions that $P_{b,L}(\overline{\gamma})$ and $P_{b,U}(\overline{\gamma})$ are continuous and strictly decreasing in $\overline{\gamma}$. Therefore the inverse functions $P_{b,L}^{-1}(P_b^\star)$ and $P_{b,U}^{-1}(P_b^\star)$ exist and have unique solutions for each $P_b^\star \in [0, S(N)] = [0, P_{b,L}(0)] \subset [0, P_{b,U}(0)]$. Hence, $\overline{\gamma}_L^\star = P_{b,L}^{-1}$ and $\overline{\gamma}_U^\star = P_{b,U}^{-1}$ can be obtained explicitly by inverting (5), (3) and (4) to get (7a) and (7b). The fact that $P_{b,L}(\overline{\gamma})$, $P_b(\overline{\gamma})$ and $P_{b,U}(\overline{\gamma})$ are continuous and strictly decreasing and $P_{b,L} \leq P_b \leq P_{b,U}$ implies that $\overline{\gamma}_L^\star \leq \overline{\gamma}^\star \leq \overline{\gamma}_U^\star$.

III. Bounds on the BIT Error Outage

For analog communication systems outage probability is typically defined in the SNR sense; that is, the probability that the mean SNR (averaged over fast fading) falls below a minimum acceptable value, called the target SNR. The BEO defined here as the probability that BEP exceed a maximum tolerable level is more appropriate for digital mobile radio. We consider mobile radio applications where $\overline{\gamma}$ varies, due to for example shadowing, at a rate much slower than Rayleigh fading [2]. Thus, the BEO is defined as

$$P_o(P_b^\star) = \mathbb{P}\left\{\frac{1}{\pi}\int_0^\Theta \left(\frac{\sin^2\theta}{\sin^2\theta + \overline{\gamma}}\right)^N d\theta \geq P_b^\star\right\}. \tag{8}$$

Since the BEP decreases with $\overline{\gamma}$, (8) is equivalent to

$$P_o(P_b^\star) = \int_0^{P_b^{-1}(P_b^\star)} p_{\overline{\gamma}}(\xi)\, d\xi \tag{9}$$

where $p_{\overline{\gamma}}(\xi)$ is the probability density function (pdf) of $\overline{\gamma}$. Analysis of (9) requires the inverse BEP expression and the bounds in (6) and (7) on the inverse BEP can be used to obtain the bounds on (9). Note that $p_{\overline{\gamma}}(\xi)$ in (9) is nonnegative, and hence we obtain the lower and the upper bounds on the P_o for coherent detection of BPSK signals as

$$P_{o,L}(P_b^\star) \leq P_o(P_b^\star) \leq P_{o,U}(P_b^\star) \tag{10}$$

where

$$P_{o,L}(P_b^\star) = \int_0^{\overline{\gamma}_L^\star} p_{\overline{\gamma}}(\xi)\, d\xi \tag{11a}$$

$$P_{o,U}(P_b^\star) = \int_0^{\overline{\gamma}_U^\star} p_{\overline{\gamma}}(\xi)\, d\xi. \tag{11b}$$

A. Log-Normal Distributed Shadowing

Now we consider the case of a shadowing environment in which $\overline{\gamma}$ is log-normal distributed with parameters μ_{dB} and σ_{dB}^2 (i.e., $\overline{\gamma}_{\text{dB}} = 10\ \log_{10}\overline{\gamma}$ is a Gaussian r.v. with mean μ_{dB} and variance σ_{dB}^2) [2]. Since the logarithm is monotonic, $P_o(P_b^\star)$ is lower and upper bounded, respectively, by

$$P_{o,L} = Q\left(\frac{\mu_{\text{dB}} - 10\log_{10}\left[\overline{\gamma}_L^\star(P_b^\star)\right]}{\sigma_{\text{dB}}}\right) \tag{12}$$

$$P_{o,U} = Q\left(\frac{\mu_{\text{dB}} - 10\log_{10}\left[\overline{\gamma}_U^\star(P_b^\star)\right]}{\sigma_{\text{dB}}}\right). \tag{13}$$

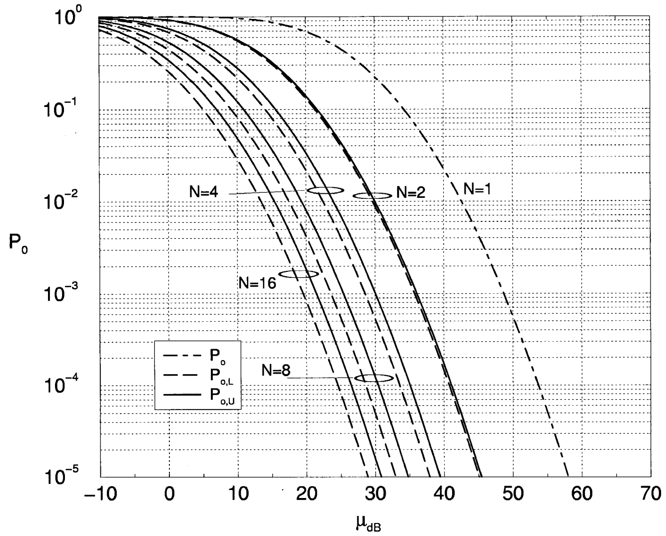

Fig. 2. $P_{o,L}$ and $P_{o,U}$ versus $\mu_{\rm dB}$ for BPSK with $\sigma_{\rm dB} = 8$, $P_b^{\star} = 10^{-3}$ and $N = 2\,4,\ 8,\ 16$. The exact P_o for $N = 1$ is also shown.

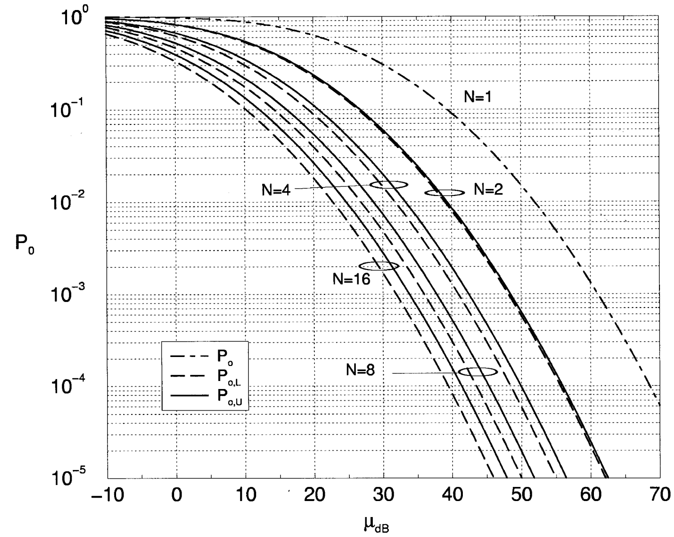

Fig. 3. $P_{o,L}$ and $P_{o,U}$ versus $\mu_{\rm dB}$ for BPSK with $\sigma_{dB} = 12$, $P_b^{\star} = 10^{-3}$ and $N = 2,\ 4,\ 8,\ 16$. The exact P_o for $N = 1$ is also shown.

IV. Numerical Results

In this section we evaluate the lower and upper bound on the BEO for the case of log-normal distributed shadowing using (12) and (13). Fig. 2 shows $P_{o,L}$ and $P_{o,U}$ versus $\mu_{\rm dB}$, both for coherent detection of BPSK with $\sigma_{\rm dB} = 8$ for $P_b^{\star} = 10^{-3}$ and $N = 1,\ 2,\ 4,\ 8,$ and 16. Note that for the case of $N = 1$ (i.e., without antenna diversity), the exact BEO can be derived. In fact, since in this case the BEP $P_b(\overline{\gamma}) = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{1+\overline{\gamma}}}\right)$ is invertible, we derive the following expression for the BEO

$$P_o\left(P_b^{\star}\right) = Q\left(\frac{\mu_{\rm dB} - 10\log_{10}\left[\frac{(1-2P_b^{\star})^2}{4P_b^{\star}(1-P_b^{\star})}\right]}{\sigma_{\rm dB}}\right). \qquad (14)$$

In general, by using (12) and (13) the improvement in terms of BEO due to MRC as a function of the diversity order is quantified.

For a fixed P_o, we now can obtain lower and upper bounds on the requirement on the parameter $\mu_{\rm dB}$ corresponding to the median value of the shadowing level. This is useful for the design of digital radio systems with diversity reception. For example, the maximum distance of the radio-link or the cluster-size for cellular systems can be estimated when the path-loss law is known.

Similar results are given in Fig. 3 with a required $P_b^{\star} = 10^{-3}$ and $\sigma_{\rm dB} = 12$ (e.g., for indoor communications) and $N = 1,\ 2,\ 4,\ 8,$ and 16.

V. Conclusions

In this work, new lower and upper bounds on the BEO have been derived from a new lower and known upper bounds on the inverse BEP, respectively, for multichannel reception with MRC and BPSK modulation. As an example of application to digital mobile radio, the BEO in a log-normal shadowing environment was analyzed. By using our results, we quantify the impact of the diversity order and shadowing parameters in terms of BEO. The results are useful for the design of digital radio systems with diversity in shadowing environments.

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