Efficient Evaluation of Exact Error Probability for Optimum Combining of M-ary PSK Signals

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Abstract— **In this paper, we derive an exact expression for the symbol error probability (SEP) for coherent detection of M-ary PSK signals using array of antennas with optimum combining in a Rayleigh fading environment. The proposed analytical framework is based on the theory of orthogonal polynomials and we give an effective technique to derive the SEP involving only one integral with finite integration limits. The result is general and valid for an arbitrary number of receiving antennas or co-channel interferers.**

I. Introduction

Adaptive arrays can significantly improve the performance of wireless communication systems by weighting and combining the received signals to reduce fading effects and suppress interference. In particular, with optimum combining (OC) the received signals are weighted and combined to maximize the output signal-to-interference-plus-noise ratio (SINR).

Closed-form expressions for the bit error probability (BEP) have been derived for the single interferer case under the assumption of Rayleigh fading for the desired signal in [1]. BEP expressions with Rayleigh fading of the desired signal and a single interferer are given in [2].

With multiple interferers of arbitrary power, Monte Carlo simulation has been used to determine the BEP [1]. To avoid Monte Carlo simulation approximations have been presented in [3, 4] for the case of equal-power interferers. However, the approximation of [3] still requires Monte Carlo simulation to obtain mean eigenvalues (a table is provided in [3] for some cases), and the approximation of [4] has been proposed when the number of interferers is less then the number of antenna elements. In [5], upper bounds on the BEP of optimum combining were derived given the average power of the interferers. Unfortunately, these bounds are generally not tight. Recently a tighter bound based on different approaches and Laguerre polynomials have been derived in [6], in the context of multiple-input multiple-output (MIMO) systems [7].

In this paper, starting from the eigenvalue distribution of Wishart complex matrices, we first derive the SEP expression for coherent detection of M-ary PSK using OC in the presence of multiple uncorrelated equal-power interferers as well as thermal noise in a flat Rayleigh fading environment. However, this requires the evaluation of multiple integrals, with the number of integral depending on the minimum of the number of antennas and interferers. To alleviate this problem, we develop an efficient method to derive the SEP. Our new approach, based on a classical technique involving orthogonal systems, leads to exact solutions that require only the evaluation of a single integral with finite limits.

M. Chiani and A. Zanella were partially supported by CNR and MIUR (Italy).

II. System Description

We consider optimum combining of multiple received signals in flat fading environment with coherent demodulation, where the fading rate is assumed to be much slower than the symbol rate. Throughout the paper $(\cdot)^T$ is the transposition operator, and $(\cdot)^\dagger$ stands for conjugation and transposition. The received signal at the N_A-element array output consists of desired signal, N_I interfering signals, and thermal noise. After matched filtering and sampling at the symbol rate, the array output vector at time k can be written as:

$$\mathbf{z}(k) = \sqrt{E_D}\mathbf{c}_D b_0(k) + \mathbf{z}_{IN}(k), \tag{1}$$

with the interference plus noise term

$$\mathbf{z}_{IN}(k) = \sqrt{E_I}\sum_{j=1}^{N_I} \mathbf{c}_{I,j} b_j(k) + \mathbf{n}(k), \tag{2}$$

where E_D and E_I are the mean (over fading) energies of the desired and interfering signal, respectively; $\mathbf{c}_D = [c_{D,1}, ..., c_{D,N_A}]^T$ and $\mathbf{c}_{I,j} = [c_{I,j,1}, ..., c_{I,j,N_A}]^T$ are the desired and j^{th} interference propagation vectors, respectively; $b_0(k)$ and $b_j(k)$ are the desired and interfering data samples, respectively, and $\mathbf{n}(k)$ represents the additive noise. We model $\mathbf{c}_D$ and $\mathbf{c}_{I,j}$ as multivariate complex-valued Gaussian vectors having $\mathbb{E}\{\mathbf{c}_D\} = \mathbb{E}\{\mathbf{c}_{I,j}\} = \mathbf{0}$ and $\mathbb{E}\left\{\mathbf{c}_D\mathbf{c}_D^\dagger\right\} = \mathbb{E}\left\{\mathbf{c}_{I,j}\mathbf{c}_{I,j}^\dagger\right\} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The interfering data samples, $b_j(k)$ for $j = 1, \ldots, N_I$, can be modeled as uncorrelated zero-mean random variables, and without loss of generality $b_0(k)$ and $b_j(k)$ are assumed to have unit variance.

The additive noise is modeled as a white Gaussian random vector with independent and identically distributed (i.i.d.) elements with $\mathbb{E}\{\mathbf{n}(k)\} = \mathbf{0}$ and $\mathbb{E}\{\mathbf{n}(k)\mathbf{n}^\dagger(k)\} = N_0\mathbf{I}$, where $N_0/2$ is the two-sided thermal noise power spectral density per antenna element.

In the following, $\mathbf{R}$ denotes the short-term covariance matrix of the disturb $\mathbf{z}_{IN}(k)$, conditioned an all interference propagation vectors, given by

$$\mathbf{R} = \mathbb{E}_{\mathbf{n},b_j(k)}\{\mathbf{z}_{IN}(k) \cdot \mathbf{z}_{IN}(k)^\dagger\} \tag{3}$$

and $\mathbb{E}_X\{\cdot\}$ denotes expectation with respect to X.

The (maximum) SINR at the output of the N_A-element array with OC can therefore be expressed as [1]

$$\gamma = E_D\mathbf{c}_D^\dagger\mathbf{R}^{-1}\mathbf{c}_D, \tag{4}$$

where it is important to remark that $\mathbf{R}$, and consequently also the SINR γ, varies at the fading rate.

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The matrix $\mathbf{R}^{-1}$ can be conveniently expressed as $\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^\dagger$ where $\mathbf{U}$ is a unitary matrix and $\mathbf{\Lambda}$ is a diagonal matrix whose elements on the principal diagonal are the eigenvalues of $\mathbf{R}$, denoted by $(\lambda_1, \ldots, \lambda_{N_A})$. Hence, the SINR given in (4) can be rewritten as:

$$\gamma = E_D \mathbf{c}_D^\dagger \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^\dagger \mathbf{c}_D = E_D \sum_{i=1}^{N_A} \frac{|u_i|^2}{\lambda_i}. \tag{5}$$

The vector $\mathbf{u} = \mathbf{U}^\dagger \mathbf{c}_D = [u_1, ..., u_{N_A}]^T$ has the same distribution as $\mathbf{c}_D$, since $\mathbf{U}$ represents a unitary transformation.

Then, it is convenient to write the matrix $\mathbf{R}$ as

$$\mathbf{R} = E_I \tilde{\mathbf{R}} + N_0 \mathbf{I}, \tag{6}$$

where $\tilde{\mathbf{R}} = \mathbf{C}_I \mathbf{C}_I^\dagger$ is an $(N_A \times N_A)$ random matrix where $\mathbf{C}_I$ defined by

$$\mathbf{C}_I \triangleq \begin{bmatrix} | & | & & | \\ \mathbf{c}_{I,1} & \mathbf{c}_{I,2} & \ldots & \mathbf{c}_{I,N_I} \\ | & | & & | \end{bmatrix}, \tag{7}$$

is an $(N_A \times N_I)$ matrix composed of N_I interference propagation vectors as columns. The eigenvalues of $\mathbf{R}$ can be written in terms of eigenvalues of $\tilde{\mathbf{R}}$, denoted by $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_A})$, as

$$\lambda_i = E_I \tilde{\lambda}_i + N_0 \qquad i = 1, \ldots, N_A, \tag{8}$$

and therefore the SINR given in (5) becomes:

$$\gamma = E_D \sum_{i=1}^{N_A} \frac{|u_i|^2}{E_I \tilde{\lambda}_i + N_0}. \tag{9}$$

Note that the eigenvalues vary at the fading rate.

We now investigate the statistical properties of $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_A})$; this is a problem regarding the eigenvalues distribution of complex Wishart matrices [8]. By using some results of [9] dealing with the *ordered* eigenvalues $\tilde{\boldsymbol{\lambda}} = [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_{\min}}]^T$, it is straight forward to show that the joint p.d.f. of the first $N_{\min} \triangleq \min\{N_A, N_I\}$ *unordered* eigenvalues of $\tilde{\mathbf{R}}$ is N_I.

$$f_{\tilde{\boldsymbol{\lambda}}}(x_1, \ldots, x_{N_{\min}}) = \frac{K}{N_{\min}!}\left[\prod_{i=1}^{N_{\min}} e^{-x_i} x_i^{N_{\max}-N_{\min}}\right] \cdot \prod_{i=1}^{N_{\min}-1}\left[\prod_{j=i+1}^{N_{\min}} (x_i - x_j)^2\right], \tag{10}$$

where $N_{\max} \triangleq \max\{N_A, N_I\}$ and K is the normalizing constant given by $K = \frac{\pi^{N_{\min}(N_{\min}-1)}}{\tilde{\Gamma}_{N_{\min}}(N_{\max})\tilde{\Gamma}_{N_{\min}}(N_{\min})}$, with $\tilde{\Gamma}_{N_{\min}}(N_{\max}) = \pi^{N_{\min}(N_{\min}-1)/2}\prod_{i=1}^{N_{\min}}(N_{\max} - i)!$. The additional $N_A - N_{\min}$ eigenvalues of $\tilde{\mathbf{R}}$ are identically equal to zero.

III. Derivation of the Symbol Error Probability

The SEP for optimum combining in the presence of multiple co-channel interferers and thermal noise in a fading environment can be written as [9]

$$P_e = \mathbb{E}_{\tilde{\boldsymbol{\lambda}}}\left\{P_{e|\tilde{\boldsymbol{\lambda}}}\right\} = \int_0^\infty \cdots \int_0^\infty P_{e|\tilde{\boldsymbol{\lambda}}}(\boldsymbol{x}) \cdot f_{\tilde{\boldsymbol{\lambda}}}(\boldsymbol{x})\, dx_1 \ldots dx_{N_{\min}}, \tag{11}$$

where $P_{e|\tilde{\boldsymbol{\lambda}}}$ represents the SEP conditioned on the random vector $\tilde{\boldsymbol{\lambda}}$; note that now the ranges of integrals start from 0 since we are considering the p.d.f. of unordered eigenvalues. Moreover, in [9] it is shown that

$$P_{e|\tilde{\boldsymbol{\lambda}}}(\tilde{\boldsymbol{\lambda}}) = \frac{1}{\pi}\int_0^\Theta A(\theta) \prod_{i=1}^{N_{\min}} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{E_D}{E_I \tilde{\lambda}_i + N_0}}\right] d\theta, \tag{12}$$

where $c_{\text{MPSK}} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$ and

$$A(\theta) = \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}}\frac{E_D}{N_0}}\right]^{N_A - N_{\min}} \tag{13}$$

Expression (11) is exact and valid for arbitrary numbers of antennas and interferers; however, it requires the evaluation of $N_{\min}$-fold integrals, which can be cumbersome to evaluate for large $N_{\min}$. We will show how this analytical difficulty can be avoided using the properties relating to the Vandermonde matrix.

IV. Efficient Evaluation of SEP for OC

We first note that the term $\prod_{i=1}^{N_{\min}-1}\left[\prod_{j=i+1}^{N_{\min}}(x_i - x_j)\right]$ in (11) can also be seen as the determinant of the Vandermonde matrix $\mathbf{V}(x_1, \ldots, x_{N_{\min}})$ given by

$$\mathbf{V}(x_1, \ldots, x_{N_{\min}}) \triangleq \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{N_{\min}} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{N_{\min}}^{N_{\min}-1} \end{pmatrix}. \tag{14}$$

Therefore, the p.d.f. can also be written as

$$f_{\tilde{\boldsymbol{\lambda}}}(x_1, \ldots, x_{N_{\min}}) = \frac{K}{N_{\min}!}\left|\mathbf{V}(x_1, \ldots, x_{N_{\min}})\right|^2 \cdot \prod_{i=1}^{N_{\min}} e^{-x_i} x_i^{N_{\max}-N_{\min}}. \tag{15}$$

For what follows it is convenient to introduce the function[1]

$$\psi(\tilde{\lambda}, \theta) \triangleq \frac{\tilde{\lambda} + \frac{N_0}{E_I}}{\tilde{\lambda} + \frac{N_0}{E_I} + \frac{c_{\text{MPSK}} E_D}{E_I \sin^2\theta}}, \tag{16}$$

and using (15), the expression (11) becomes

$$P_e = \frac{K}{N_{\min}!}\frac{1}{\pi}\int_0^\Theta A(\theta)\int_0^\infty \cdots \int_0^\infty \left|\mathbf{V}(x_1, \ldots, x_{N_{\min}})\right|^2 \cdot \left[\prod_{i=1}^{N_{\min}} \psi(x_i, \theta) x_i^{N_{\max}-N_{\min}} e^{-x_i} dx_i\right] d\theta. \tag{17}$$

The evaluation of (17) is difficult because the integrand does not factor and the dimension of integral depends on the minimum of the number of antennas and interferers. We now give an efficient method to reduce the SEP to a single integral with finite limits. The approach is based on a classical technique commonly used in mathematical physics involving orthogonal systems.

[1] The dependence of the parameters E_D, E_I and N_0 is suppressed to simplify the notation.

Let us first consider a more general problem of evaluating

$$\mathbb{E}_{\tilde{\lambda}}\left\{\prod_{i=1}^{N_{\min}} z(\tilde{\lambda}_i,\theta)\right\} \tag{18}$$

where $z(x,\theta) \geq 0$ on every subset of the support of $\tilde{\lambda}_i$ with positive measure, and the average is over the distribution of the eigenvalues given by (15) . This problem can be efficiently solved by using some classical results from orthogonal polynomials as follows.

For each $\theta \in [0,\Theta]$, let $\mathcal{P}_\theta^{N_{\min}}$ be the space of all polynomials with degree less than or equal to $N_{\min}-1$ with measure

$$d\mu_\theta(x) = z(x,\theta)x^{N_{\max}-N_{\min}}e^{-x}dx\,, \tag{19}$$

equipped with the inner product and norm defined respectively by

$$\langle f,g\rangle(\theta) \triangleq \int_0^\infty f(x)g(x)\; z(x,\theta)x^{N_{\max}-N_{\min}}e^{-x}dx \tag{20}$$

$$\|f\|_\theta^2 \triangleq \int_0^\infty f(x)f(x)\; z(x,\theta)x^{N_{\max}-N_{\min}}e^{-x}dx \tag{21}$$

Since $z(x,\theta) \geq 0$ on every subset of the support of $\tilde{\lambda}_i$ positive measure, so is $z(x,\theta)x^{N_{\max}-N_{\min}}e^{-x} > 0$, and hence the elements $1, x, x^2, \ldots, x^{N_{\min}-1}$ of the Hilbert Space $\mathcal{P}_\theta^{N_{\min}}$ are linearly independent. This implies that there exists an orthogonal system $\{\phi_n(x,\theta)\}_{n=0}^{N_{\min}-1}$ with

$$\phi_n(x,\theta) = \phi_{n,0}(\theta) + \phi_{n,1}(\theta)\,x + \cdots + \phi_{n,n}(\theta)\,x^n, \tag{22}$$

such that $\langle\phi_n,\phi_m\rangle(\theta) = \|\phi_n\|_\theta^2\;\delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta function defined by

$$\delta_{n,m} \triangleq \begin{cases} 1 & n = m, \\ 0 & n \neq m. \end{cases} \tag{23}$$

The orthogonal system $\{\phi_n(x,\theta)\}_{n=0}^{N_{\min}-1}$ can be obtained by a Gram-Schmidt procedure using the measure $\mu_\theta(x)$ as shown in Appendix A.

Theorem 1:

$$\mathbb{E}_{\tilde{\lambda}}\left\{\prod_{i=1}^{N_{\min}} z(\tilde{\lambda}_i,\theta)\right\} = K\prod_{n=0}^{N_{\min}-1}\|\phi_n\|_\theta^2 \tag{24}$$

where K has been already defined and $\prod_{n=0}^{N_{\min}-1}\|\phi_n\|_\theta^2$ is the product norm squares of all the elements in a particular orthogonal system generated by $\mu_\theta(x)$.

Proof:

$$\mathbb{E}_{\tilde{\lambda}}\left\{\prod_{i=1}^{N_{\min}} z(\tilde{\lambda}_i,\theta)\right\} = \frac{K}{N_{\min}!}\int_0^\infty\cdots\int_0^\infty |\mathbf{V}(x_1,\ldots,x_{N_{\min}})|^2\cdot \cdot\prod_{i=1}^{N_{\min}} z(x_i,\theta)x_i^{N_{\max}-N_{\min}}e^{-x_i}dx_i \tag{25}$$

For any given $\theta \in [0,\Theta]$, the Vandermonde matrix $\mathbf{V}(x_1,\ldots,x_{N_{\min}})$ can be transformed, using the orthogonal system generated by $\mu_\theta(x)$, into $\tilde{\mathbf{V}}(x_1,\ldots,x_{N_{\min}};\theta)$ defined by

$$\tilde{\mathbf{V}}(x_1,\ldots,x_{N_{\min}};\theta) \triangleq \begin{pmatrix} \phi_0(x_1,\theta) & \cdots & \phi_0(x_{N_{\min}},\theta) \\ \phi_1(x_1,\theta) & \cdots & \phi_1(x_{N_{\min}},\theta) \\ \vdots & \ddots & \vdots \\ \phi_{N_{\min}-1}(x_1,\theta) & \cdots & \phi_{N_{\min}-1}(x_{N_{\min}},\theta) \end{pmatrix} \tag{26}$$

by means of elementary row operations, more precisely, successively subtracting linear combination of its rows from another row. Since the determinant is invariant to such row operations

$$|\mathbf{V}(x_1,\ldots,x_{N_{\min}})| = \left|\tilde{\mathbf{V}}(x_1,\ldots,x_{N_{\min}};\theta)\right|. \tag{27}$$

We now let $\mathcal{S}_{N_{\min}}$ be the set of all permutations of integers $\{0,1,\ldots,N_{\min}-1\}$, and let $\sigma \in \mathcal{S}_{N_{\min}}$ denote the particular function $\sigma : (0,1,\ldots,N_{\min}-1) \to (\sigma_1,\sigma_2,\ldots,\sigma_{N_{\min}})$ which permutes the integers $\{0,1,\ldots,N_{\min}-1\}$. The determinant can be written as

$$\left|\tilde{\mathbf{V}}(x_1,\ldots,x_{N_{\min}})\right| = \sum_{\sigma\in\mathcal{S}_{N_{\min}}} \operatorname{sgn}\{\sigma\}\prod_{i=1}^{N_{\min}}\phi_{\sigma_i}(x_i,\theta)\,, \tag{28}$$

where

$$\operatorname{sgn}\{\sigma\} = \begin{cases} +1 & \text{for even permutation,} \\ -1 & \text{for odd permutation.} \end{cases} \tag{29}$$

Substituting (27) and (28) into (25) gives

$$\mathbb{E}_{\tilde{\lambda}}\left\{\prod_{i=1}^{N_{\min}} z(\tilde{\lambda}_i,\theta)\right\} = \frac{K}{N_{\min}!}\sum_{\alpha\in\mathcal{S}_{N_{\min}}}\sum_{\sigma\in\mathcal{S}_{N_{\min}}}\operatorname{sgn}\{\alpha\}\operatorname{sgn}\{\sigma\}\cdot \prod_{i=1}^{N_{\min}}\langle\phi_{\alpha_i}(x_i,\theta),\phi_{\sigma_i}(x_i,\theta)\rangle\,. \tag{30}$$

Using the fact that $\{\phi_n(x,\theta)\}_{n=0}^{N_{\min}-1}$ are orthogonal, (30) becomes

$$\mathbb{E}_{\tilde{\lambda}}\left\{\prod_{i=1}^{N_{\min}} z(\tilde{\lambda}_i)\right\} = K\prod_{n=0}^{N_{\min}-1}\|\phi_n\|_\theta^2\,,$$

This completes the proof of Theorem 1. ■

Using Theorem 1, we immediately obtain the following theorem.

Theorem 2: The SEP for coherent detection of M-ary PSK signals using optimum combining with an N_{A}-element antenna array in the presence of N_{I} uncorrelated equal-power co-channel interferers and thermal noise in Rayleigh fading is given by

$$P_e = \frac{K}{\pi}\int_0^\Theta A(\theta)C(\theta)\;d\theta\,, \tag{31}$$

where $A(\theta)$ is given by (13) and $C(\theta) \triangleq \prod_{n=0}^{N_{\min}-1}\|\phi_n\|_\theta^2$, with $N_{\min} = \min\{N_{\mathrm{A}},N_{\mathrm{I}}\}$, is the product norm squared of all the elements in a particular orthogonal system generated by $\mu_\theta(x)$ of (19) using $z(x,\theta) = \psi(x,\theta)$.

Thus the derivation of the SEP for coherent detection of M-ary PSK using OC, involving the N-fold integrals in (11), essentially reduces to a simple single integral over θ with finite limits. The integrand is a product of two functions $A(\theta)$ and $C(\theta)$; the former function $A(\theta)$ involves trigonometric functions and is given by (13) and the latter function $C(\theta)$ can be evaluated easily based on the approach illustrated in Appendix A. Finally, the SEP expression (31) can be efficiently and rapidly evaluated using standard mathematical packages, even for large number of antennas and/or co-channel interferers, where previous studies relied on highly time-expensive simulations.

V. Numerical Results

In this section the performance in terms of SEP of adaptive arrays with OC is investigated by the analytical approach given in Theorem 2, with different choices of the signal to noise ratio (SNR) defined as E_D/N_0, the ratio between the desired received signal power and the total interfering power (SIR) defined as $E_D/(N_I \cdot E_I)$, the number of interferers, and the number of antennas.

Fig. 1 shows the SEP as a function of SNR, when the number of antenna branches has been fixed to $N_A = 6$, with $N_I = 4$ interfering signals and SIR=10 dB. Several modulation formats are considered: BPSK, QPSK, 8-PSK, 16-PSK and 32-PSK. In the figure are also shown some semi-analytical results, obtained by generating the random propagation vectors, computing the SINR by (4) and then the error probability by means of [9, eq. 17]. Since the analytical framework proposed in this work provides an exact result in the same hypotheses, we find perfect agreement between analysis and semi-analytical results. Moreover, the comparison between different modulation formats shows that if we fix, for example, a target SEP at 10^{-3}, BPSK requires a SNR of about 2 dB, and this value rises to about 6 dB with QPSK, and more than 14 dB with higher level formats.

Fig. 2 shows the SEP with 5 antenna branches as a function of SNR when the SIR is fixed to 10 dB and coherent detection of 8-PSK is considered. The number of interfering signals ranges from 1 to 8. The figure shows that when the number of interferers becomes equal to or larger than the number of receiving antennas, the curve exhibits an error floor. This can be easily explained by remembering that adaptive array systems have $N_A - 1$ degrees of freedom to cope with interfering signals and thermal noise. When the number of interfering is greater than the array degrees of freedom, the system is not able to null out the interferers and, for large values of SNR, the performance is limited by the interfering power. The opposite is true when the number of antenna branches is greater than the number of interferers; here the additional $L_{Div} = N_A - N_I$ degrees of freedom are used to mitigate thermal noise and desired signal multipath fadindg, and this provides an asymptotic behavior for SEP proportional to $1/(SNR)^{L_{Div}}$ (in other words, a diversity degree L_{Div} with respect to fading of the useful signal).

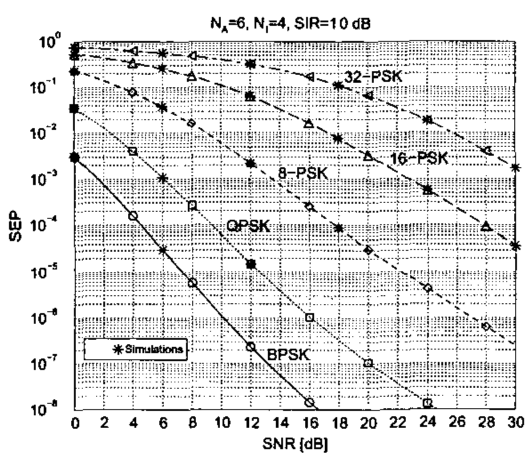

Fig. 1. The SEP as a function of SNR for N_A=6, N_I=4, SIR=10 dB, several modulation formats are considered: BPSK, QPSK, 8-PSK, 16-PSK and 32-PSK. Semi-analytical results are also provided (symbols).

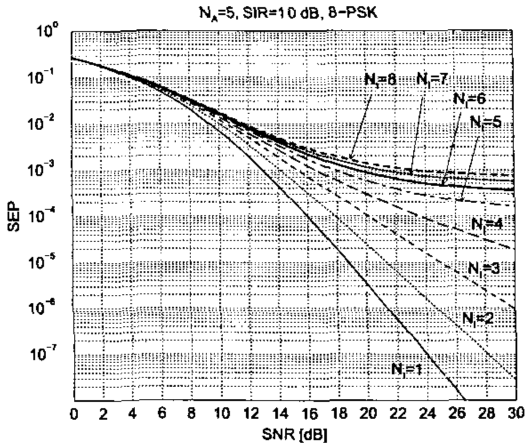

Fig. 2. The SEP as a function of SNR for N_A=5, 8-PSK, SIR=10 dB; the number of interferers ranges from 1 to 8.

Fig. 3 shows the SEP as a function of SIR with 6 antenna branches and 8-PSK. The number of interferers ranges from 2 to 6, and the SNR is varied from 5 to 10 dB. The figure shows that, when the interference power becomes almost comparable with the thermal noise power, the number of interferers does not play an important role. Moreover, the comparison between optimum combining and maximal ratio combining (MRC) shows that, as expected, when the interference power is small, the performance of the two schemes is not much different; on the other hand, when the SIR decreases, the performance with optimum combining is strongly dependent on the number of interferers and tends to be closer to that with MRC for as N_I increases.

VI. Conclusions

In this work, starting from an expression requiring the numerical evaluation of nested integrals and by using the theory of orthogonal polynomials, we obtained a simple and numerically stable solution for the exact symbol error probability with optimum combining of signals. We assumed coherent detection of M-ary PSK in the presence of multiple uncorrelated equal-power interferers and thermal noise in a flat Rayleigh fading environment. The new approach makes possible the exact SEP

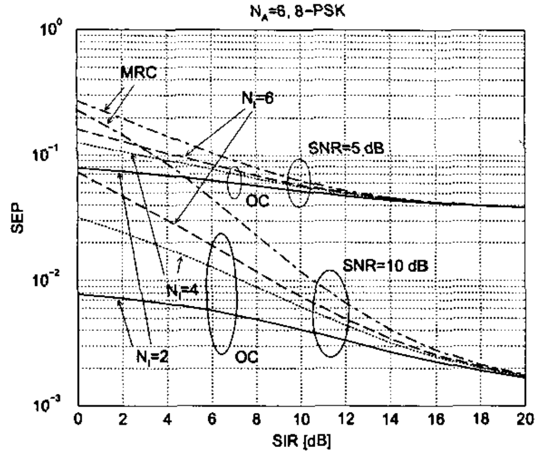

Fig. 3. The SEP as a function of SIR for N_A=6, 8-PSK, SNR=5 and 10 dB, and N_I=2, 4 and 6; comparison between OC and MRC.

evaluation for wireless systems with arbitrary number of interferers and antenna elements.

Appendix A: Derivation of the Orthogonal System

As pointed out earlier in Section IV, the polynomials $1, x, x^2, \ldots, x^{N_{\min}-1}$ in the Hilbert Space $\mathcal{P}_\theta^{N_{\min}}$, with inner product $\langle f, g\rangle(\theta)$ and norm $\|f\|_\theta$ given by (20) and (21) respectively, are linearly independent. Therefore we can apply the Gram-Schmidt procedure to obtain the orthogonal systems as follows.

The normalization of the polynomial 1 gives the function $\phi_0(x,\theta)$ as follows

$$1 \longrightarrow \phi_0(x,\theta) = 1\,. \tag{32}$$

The polynomial x produces the second function by

$$x \longrightarrow \phi_1(x,\theta) = x - \frac{\langle x, \phi_0(x,\theta)\rangle}{\|\phi_0\|_\theta^2}\, \phi_0(x,\theta)\,. \tag{33}$$

In general, the polynomial x^n for $n = 0, \ldots, N_{\min} - 1$ transforms into

$$x^n \longrightarrow \phi_n(x,\theta) = x^n - \sum_{j=0}^{n-1} \frac{\langle x^n, \phi_j(x,\theta)\rangle}{\|\phi_j\|_\theta^2}\, \phi_j(x,\theta)\,. \tag{34}$$

Now, adopting the following notation for polynomials

$$\phi_n(x,\theta) = \phi_{n,0}(\theta) + \phi_{n,1}(\theta)\, x + \cdots + \phi_{n,n}(\theta)\, x^n, \tag{35}$$

the norm square of $\phi_n(x,\theta)$ can be expressed as

$$\|\phi_n\|_\theta^2 = \sum_{l=0}^{n}\sum_{m=0}^{n} \phi_{n,l}(\theta)\phi_{n,m}(\theta)\, \left\langle x^l, x^m \right\rangle\,. \tag{36}$$

Using the inner product (20) with $z(x,\theta) = \psi(x,\theta)$, $\|\phi_n\|_\theta^2$ becomes

$$\|\phi_n\|_\theta^2 = \sum_{l=0}^{n}\sum_{m=0}^{n} \phi_{n,l}(\theta)\phi_{n,m}(\theta)\, G_{l+m}(\theta)\,. \tag{37}$$

where

$$G_k(\theta) \triangleq \int_0^\infty x^{k+N_{\max}-N_{\min}} e^{-x} \psi(x,\theta)\, dx\,. \tag{38}$$

A closed form expression for $G_k(\theta)$ can be derived as [10, eq. 3.353.5]

$$\begin{aligned} G_{k-N_{\max}+N_{\min}}(\theta) &= \zeta(\theta)^k e^{\zeta(\theta)} k! \left[\zeta(\theta)\,(1+k) + \right. \\ &\quad \left. + \Gamma\left(-1-k, \zeta(\theta)\right) \frac{N_0}{E_I}\, \Gamma\left(-k, \zeta(\theta)\right)\right] \end{aligned} \tag{39}$$

where $\zeta(\theta) = \frac{c_{\text{MPSK}} E_D}{E_I \sin^2\theta} + \frac{N_0}{E_I}$.

The coefficients $\phi_{n,m}(\theta)$ can be calculated iteratively using the following formula, which we derive as follows. Substituting (35) into (34) and using the inner product (20) with $z(x,\theta) = \psi(x,\theta)$ we have

$$\phi_n(x,\theta) = x^n - \sum_{m=0}^{n-1}\left[\sum_{j=m}^{n-1} \frac{1}{\|\phi_j\|_\theta^2}\phi_{j,m}(\theta) \sum_{k=0}^{j} \phi_{j,k}(\theta)\, G_{n+k}(\theta)\right] x^m\,. \tag{40}$$

Comparing (35) and (40), we obtain the $m^{\underline{\text{th}}}$ coefficient of the $n^{\underline{\text{th}}}$ polynomial as

$$\phi_{n,m}(\theta) = -\sum_{j=m}^{n-1} \frac{1}{\|\phi_j\|_\theta^2}\phi_{j,m}(\theta) \sum_{k=0}^{j} \phi_{j,k}(\theta)\, G_{n+k}(\theta), \tag{41}$$

with $\phi_{n,n}(\theta) = 1$ and $m = 0, \ldots, n-1$.

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