On the SNR Penalty of MPSK With Hybrid Selection/Maximal Ratio Combining Over i.i.d. Rayleigh Fading Channels

Moe Z. Win, *Senior Member, IEEE*, Norman C. Beaulieu, *Fellow, IEEE*, Lawrence A. Shepp, Benjamin F. Logan, Jr., and Jack H. Winters, *Fellow, IEEE*

***Abstract*—Closed-form expressions that lower and upper bound the penalty of hybrid selection/maximal ratio combining relative to maximal ratio combining (MRC) for M-ary phase-shift keying (MPSK) modulations are proved. The bounds offer simple-to-evaluate explicit expressions, and are typically within 0.6 dB for hybrid systems with diversity order up to eight that use at least two branches, yet are independent of signal-to-noise ratio (SNR). Contrary to conclusions conjectured in a recently published paper, it is proved that the SNR penalty is not a constant, independent of SNR. It is also shown that previous estimates of the performance losses of selection diversity relative to MRC underestimate or lower bound the losses for MPSK modulation systems, and that the true loss can be significantly larger than previously believed. An upper bound to this loss is also obtained.**

***Index Terms*—Diversity combining, error probability, fading channels, hybrid selection/maximal ratio combining (H-S/MRC), maximal ratio combining (MRC), selection diversity (SD).**

I. Introduction

PRACTICAL considerations of diversity systems with reduced complexity for wireless communications have given impetus to hybrid selection/maximal ratio combining (H-S/MRC) techniques [1]–[9]. In H-S/MRC, the receiver selects the L branches (from N available diversity branches) with largest signal-to-noise ratios (SNRs) for maximal ratio combining (MRC), offering complexity reduction with good performance and bridging the performance gap between selection diversity (SD) and MRC. From a system design point of view, it is useful to quantify the tradeoff between reduction in complexity and loss in performance.

It is well known that the average SNR of MRC is equal to the sum of the average branch SNRs [10]. The performance of H-S/MRC is less well understood. A long and complex analysis giving the average SNR of H-S/MRC was presented in [6]. A more concise and tractable analysis, based on a "virtual branch technique," which gives the variance of the SNR as well as the average SNR, was presented in [8]. The average symbol-error probability (SEP) of digital modulation schemes using H-S/MRC was derived in [9]. However, the results require evaluation of a double or single summation, each term of which requires a single numerical integration over a finite interval.

In this paper, we derive simple lower and upper bounds for the SEP performance of H-S/MRC used with M-ary phase-shift keying (MPSK) modulation. The bounds are derived by comparing the SEP performance of H-S/MRC with that of N-branch MRC. Since H-S/MRC combines only L out of N branches, it incurs an SNR loss, or penalty, relative to MRC where all N branches are combined. The penalty is defined in an error-rate sense as the increase in SNR required for hybrid combining to achieve the same target SEP as MRC. It is to be expected that this penalty is a function of the target SEP, and hence, a function of SNR.

The SNR penalty is rigorously lower and upper bounded. The bounds are useful not only because they are simple explicit closed-form expressions, but also because they do not depend on the average branch SNR and are valid for all values of SNR. Thus, the SEP of H-S/MRC systems can be easily estimated to a high degree of accuracy (or rigorously lower and upper bounded) by using the new bounds with the wide range of previously published results on MRC with MPSK.

We first establish asymptotic analytical expressions for the SNR penalties that are incurred at small and large SNR values. In the course of obtaining these asymptotes, we prove that a conjecture stated in [11], that the SNR penalty incurred by H-S/MRC relative to MRC is a constant, independent of SNR is, surprisingly, false.

The special case of H-S/MRC with $L = 1$ is well-known selection diversity (SD). The SD method has been used for decades [10] and continues to find widespread application owing to its simplicity and low implementation cost [12]–[15]. Using the results of our analyses, we derive some interesting, previously unknown, conclusions regarding the performance of SD relative to MRC. In particular, it is shown that previous

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M. Z. Win was with the Wireless Systems Research Department, AT&T Labs - Research, Middletown, NJ 07748-4801 USA. He is now with the Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: win@ieee.org).

N. C. Beaulieu is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 2V4, Canada (e-mail: beaulieu@ee.ualberta.ca).

L. A. Shepp is with the Department of Statistics, Rutgers University, Piscataway, NJ 08855 USA (e-mail: shepp@stat.rutgers.edu).

B. F. Logan, Jr. is with Shannon Laboratories, AT&T Labs—Research, Florham Park, NJ 07932 USA (e-mail: logan@research.att.com)

J. H. Winters was with the Wireless Systems Research Department, AT&T Labs—Research, Middletown, NJ 07748-4801 USA. He is now with Jack Winters Communications, LLC, Middletown, NJ 07748-2070 USA (e-mail: jack@JackWinters.com).

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assessments underestimate or lower bound the performance losses of SD relative to MRC with MPSK modulations. Furthermore, we obtain an upper bound to this loss.

This paper is organized as follows. Section II describes the system model, recalls relevant diversity combining results needed for the paper, and defines the system parameters. The asymptotic SNR penalties are derived in Section III. Simple bounds on the SNR penalty and the SEP are presented in Section IV. Section V presents some numerical examples, and conclusions are given in Section VI. An asymptotic expansion of the SEP valid for small SNR is derived in Appendix A. Useful mathematical inequalities are derived in Appendix B, and using them, the bounds are proved in Appendix C.

II. Diversity Combining Analysis

In this section, the system model is presented. Some previous results regarding diversity, needed for the development of this paper, are also summarized.

A. Preliminaries

Throughout the paper, $\mathbb{Z}_L \triangleq \{1,2,\ldots,L\}$, $\mathbb{Z}_N \triangleq \{1,2,\ldots,N\}$, and $\mathbb{Z}_L^N \triangleq \{L+1, L+2, \ldots, N\}$. Whenever $L \geq N$, $\mathbb{Z}_L^N \triangleq \emptyset$, i.e., the empty set. For each $i \in \mathbb{Z}_N$, let γ_i denote the instantaneous SNR of the ith diversity branch defined by $\gamma_i \triangleq \alpha_i^2 E_{\mathrm{s}}/N_{0i}$, where $2E_{\mathrm{s}}$ is the average symbol energy, α_i is the instantaneous fading amplitude, and $2N_{0i}$ is the two-sided noise power spectral density of the ith branch. We consider the widely-used Rayleigh fading model for which the α_i's are independent and identically distributed (i.i.d.) Rayleigh random variables (rv's), and thus, the γ_i's are i.i.d. continuous rv's, each with exponential probability density function (pdf) and mean $\Gamma = \mathbb{E}\{\gamma_1\}$.

An H-S/MRC diversity system has instantaneous output SNR of the form

$$\gamma_{\text{H-S/MRC}} = \sum_{i \in \mathbb{Z}_L} \gamma_{[i]} \tag{1}$$

where $\gamma_{[i]}$ is the ordered γ_i, i.e., $\gamma_{[1]} > \gamma_{[2]} > \cdots > \gamma_{[N]}$, N is the number of available diversity branches, and $1 \leq L \leq N$.[1]

B. SEP of H-S/MRC and MRC

The SEP for H-S/MRC in a slowly fading multipath environment is obtained by averaging the conditional SEP over the channel ensemble as $P_e = \mathbb{E}_{\gamma_{\text{H-S/MRC}}}\{\mathbb{P}\{e|\gamma_{\text{H-S/MRC}}\}\}$. For coherent detection of MPSK, the conditional SEP, denoted by $\mathbb{P}\{e|\gamma_{\text{H-S/MRC}}\}$, is given (see, for example, [17]) by

$$\mathbb{P}\{e_{\text{MPSK}}|\gamma_{\text{H-S/MRC}}\} = \frac{1}{\pi}\int_0^{\Theta} e^{-(c_{\text{MPSK}}/\sin^2\theta)\gamma_{\text{H-S/MRC}}}\, d\theta \tag{2}$$

[1]Note that the possibility of at least two equal $\gamma_{[i]}$'s is excluded, since $\gamma_{[i]} \neq \gamma_{[j]}$ *almost surely* for continuous rv's γ_i [16].

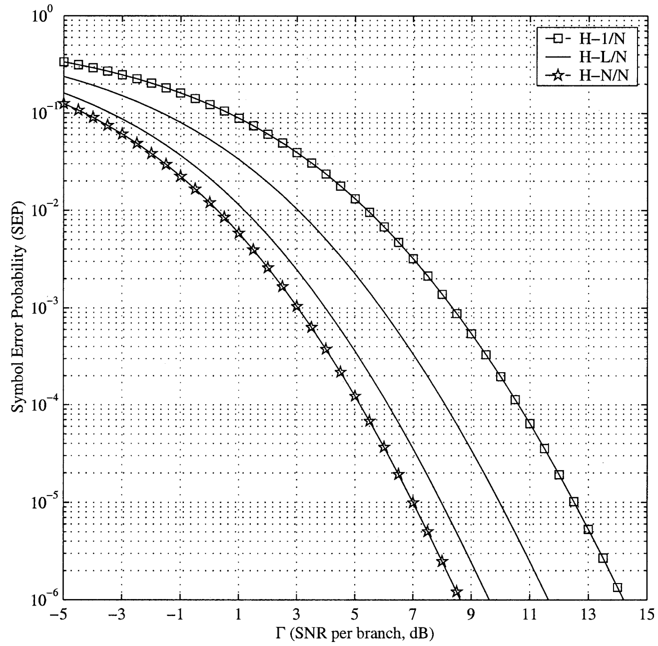

Fig. 1. SEP for coherent detection of 4-PSK with H-S/MRC as a function of average SNR per branch in decibels for $N = 8$ and various L. The curves depict $L = 1$, 2, 4, and 8 in successively lower positions.

where $c_{\text{MPSK}} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. A convenient expression for the SEP of H-S/MRC given in [9] is

$$P_{e,\text{H-S/MRC}}(\Gamma) = \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma + \sin^2\theta}\right]^L \cdot \prod_{n \in \mathbb{Z}_L^N}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma\frac{L}{n} + \sin^2\theta}\right] d\theta. \tag{3}$$

The form of (3) is particularly tractable for further analysis and we shall use it to derive the central results of this paper. Note that SD and MRC are special cases of H-S/MRC with $L = 1$ and $L = N$, respectively. Substituting $L = N$ into (3), the SEP for coherent detection of MPSK with MRC is obtained as

$$P_{e,\text{MRC}}(\Gamma) = \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{c_{\text{MPSK}}\Gamma + \sin^2\theta}\right]^N d\theta. \tag{4}$$

C. SNR Penalty

The SEP versus average SNR per branch for coherent detection of MPSK with $M = 4$ (4-PSK) using H-S/MRC is plotted in Fig. 1 for $L = 1$, 2, 4, and 8 with $N = 8$. The notation H-L/N is used to denote H-S/MRC that selects and combines L out of N branches. Note that H-1/1 is a single branch receiver, and H-$1/N$ and H-N/N are N-branch SD and MRC, respectively. Since H-S/MRC combines only L branches, it incurs an SNR loss, or penalty, relative to MRC where all N branches are combined. For a digital communication system, we define the SNR penalty as the increase in SNR required by H-S/MRC to achieve the same target SEP as N-branch MRC. That is

$$P_{e,\text{H-S/MRC}}(\beta\Gamma) = P_{e,\text{MRC}}(\Gamma) \tag{5}$$

where $P_{e,\text{H-S/MRC}}(x)$, $P_{e,\text{MRC}}(x)$, β, and Γ are the SEP of H-S/MRC at SNR x, the SEP of MRC at SNR x, the SNR penalty, and the average branch SNR, respectively. Note that the SNR penalty, in general, is a function of the target SEP, and hence, a function of the average branch SNR; that is, $\beta = \beta(\Gamma)$. Equation (5) defines $\beta(\Gamma)$ implicitly. It can be rewritten to give

$$\beta(\Gamma) = \frac{1}{\Gamma} P^{-1}_{e,\text{H-S/MRC}}\{P_{e,\text{MRC}}(\Gamma)\} \tag{6}$$

explicitly, where $P^{-1}_{e,\text{H-S/MRC}}(x)$ is the inverse H-S/MRC SEP function. Although the inverse function may be obtained numerically if we have $P_{e,\text{H-S/MRC}}(x)$ in hand, the function $\beta(\cdot)$ is not known in closed-form.

Based on limited numerical results (binary modulations with $L = 2$, $N = 3$ and 4), a plausible conjecture was made in [11] that the SNR penalty of H-S/MRC relative to MRC is a constant, independent of SNR. It is stated that "this result is obvious from the numerical results, but certainly not obvious from the analytical expressions." (The analytical expressions are not used to prove this conclusion in [11].) Consider the SEP results for H-S/MRC with H-4/8 and of MRC with H-8/8 in Fig. 1. Inspection of Fig. 1 gives credence to this thinking, as the penalty *appears* numerically to be constant; for example, the SNR penalties for SEP values of 10^{-3} and 10^{-5} are graphically the same. In the next section, we present analytic asymptotic penalties for small and large SNR for all L and N. It will be shown that though they are not equal, they are sometimes quite close, and hence, although the conjecture of [11] is rigorously false, it may be a good approximation for some cases.

III. Asymptotic SNR Penalties

Theorem 1: The asymptotic SNR penalty for small and large SNR, is given by

$$\beta_{\text{L}}^{\text{A}} = \left[\frac{\kappa(N,N)}{\kappa(L,N)}\right]^2 \tag{7a}$$

and

$$\beta_{\text{U}}^{\text{A}} = \left\{\frac{N!}{L!L^{N-L}}\right\}^{1/N} \tag{7b}$$

respectively, where $\kappa(L,N)$ is defined to be

$$\kappa(L,N) \triangleq \frac{1}{\pi}\int_0^\infty \left\{1 - \left[\frac{u^2}{1+u^2}\right]^L \prod_{n\in\mathbb{Z}_L^N}\left[\frac{u^2}{\frac{L}{n}+u^2}\right]\right\} du. \tag{8}$$

Proof [Penalty for Asymptotically Small SNR]: It can be shown, using *Lemma 1* given in Appendix A, that the asymptotic expansion for $P_{e,\text{H-S/MRC}}(\Gamma)$ and $P_{e,\text{MRC}}(\Gamma)$ for small Γ is given, respectively, by

$$P_{e,\text{MRC}}(\Gamma) \approx \frac{\Theta}{\pi} - \kappa(N,N)\Gamma^{1/2} + o(\Gamma^{1/2}) \tag{9}$$

$$P_{e,\text{H-S/MRC}}(\Gamma) \approx \frac{\Theta}{\pi} - \kappa(L,N)\Gamma^{1/2} + o(\Gamma^{1/2}) \tag{10}$$

where $\kappa(\cdot,\cdot)$ is given by (8). Note that $\kappa(L,N) < \kappa(N,N)$ for $L < N$. Since the inequality is strict (except for $L = N$, in which case they are trivially equal), a change of scale $P_{e,\beta}(\Gamma) \triangleq P_{e,\text{MRC}}(\beta^{-1}\Gamma)$ results in the two functions $P_{e,\beta}(\Gamma)$ and $P_{e,\text{H-S/MRC}}(\Gamma)$ touching asymptotically. The asymptotic SNR penalty $\beta_{\text{L}}^{\text{A}}$ is determined by the value of β such that

$$P_{e,\beta}(\Gamma) = P_{e,\text{H-S/MRC}}(\Gamma). \tag{11}$$

Substituting (9) and (10) into (11) gives (7a), which proves the first half of *Theorem 1*. □

Proof [Penalty for Asymptotically Large SNR]: Note that since $P_{e,\text{H-S/MRC}}(\Gamma)$ and $P_{e,\text{MRC}}(\Gamma)$ are both analytic, they each have a power series expansion in terms of $1/\Gamma$ about $\Gamma = \infty$. Let $b_{i,\text{H-S/MRC}}$ and $b_{i,\text{MRC}}$, respectively, denote the power series coefficients of $P_{e,\text{H-S/MRC}}(\Gamma)$ and $P_{e,\text{MRC}}(\Gamma)$ in terms of $1/\Gamma$ near $\Gamma = \infty$.

Note that the first nonzero coefficients in the power series expansion are

$$b_{N,\text{H-S/MRC}} = \left[\prod_{n\in\mathbb{Z}_L^N}\frac{n}{L}\right]\frac{1}{c_{\text{MPSK}}^N\pi}\int_0^\Theta \left[\sin^{2N}\theta\right] d\theta \tag{12}$$

and

$$b_{N,\text{MRC}} = \frac{1}{c_{\text{MPSK}}^N\pi}\int_0^\Theta \left[\sin^{2N}\theta\right] d\theta. \tag{13}$$

Since $b_{N,\text{MRC}} < b_{N,\text{H-S/MRC}}$ for $L < N$ (except for $L = N$, in which case they are trivially equal), a change of scale results in an Nth order "osculation" of the two functions $P_{e,\beta}(\Gamma)$ and $P_{e,\text{H-S/MRC}}(\Gamma)$.[2] The asymptotic penalty $\beta_{\text{U}}^{\text{A}}$ is the value of β determined by the Nth order "osculation" conditions, i.e.,

$$b_{N,\beta} = b_{N,\text{H-S/MRC}} \tag{14}$$

where $b_{n,\beta}$ denotes the nth power series coefficient of $P_{e,\beta}(\Gamma)$ in terms of $1/\Gamma$ about $\Gamma = \infty$. Equation (14) implies that

$$\frac{\left(\beta_{\text{U}}^{\text{A}}\right)^N}{c_{\text{MPSK}}^N\pi}\int_0^\Theta \left[\sin^{2N}\theta\right] d\theta = \left[\prod_{n\in\mathbb{Z}_L^N}\frac{n}{L}\right] \cdot \frac{1}{c_{\text{MPSK}}^N\pi}\int_0^\Theta \left[\sin^{2N}\theta\right] d\theta \tag{15}$$

which results in (7b). This proves the second half of *Theorem 1*. □

Interestingly, one sees from (7a) and (7b) that the asymptotic penalties are independent of M for MPSK. When $(L,N) = (2,3)$, $(\beta_{\text{L}}^{\text{A}}, \beta_{\text{U}}^{\text{A}}) = (0.5203\text{dB}, 0.5870\text{dB})$ and $\beta_{\text{L}}^{\text{A}}$ is close to $\beta_{\text{U}}^{\text{A}}$, the difference being only 0.0667 dB. However, when $(L,N) = (2,8)$ one has $(\beta_{\text{L}}^{\text{A}}, \beta_{\text{U}}^{\text{A}}) = (2.5930\text{dB}, 3.1229\text{dB})$ and the difference is 0.5299 dB, clearly demonstrating that the penalty is not constant for all values of SNR.

While $\beta_{\text{L}}^{\text{A}}$ and $\beta_{\text{U}}^{\text{A}}$ provide useful information about the performance of H-S/MRC, it is also important to assess the performance of H-S/MRC for arbitrary SNR. General results valid for arbitrary SNR are presented in the next section and proved in subsequent sections.

[2]Two curves are said to be in nth order "osculation" if their first n derivatives (including $n = 0$) are equal [18].

IV. Simple Bounds

The following theorem states simple and explicit expressions of lower and upper bounds for the SNR penalty of H-S/MRC relative to MRC with MPSK modulations.

Theorem 2: Let β_{L} and β_{U} be defined as

$$\beta_{\mathrm{L}} \triangleq \frac{N}{L\left(1+\sum_{n\in\mathbb{Z}_L^N}\frac{1}{n}\right)} \tag{16a}$$

and

$$\beta_{\mathrm{U}} \triangleq \left\{\frac{N!}{L!L^{N-L}}\right\}^{1/N} \tag{16b}$$

respectively. The SNR penalty of H-S/MRC relative to MRC is lower and upper bounded by

$$\beta_{\mathrm{L}} \le \beta(\Gamma) \le \beta_{\mathrm{U}} \tag{17}$$

for coherent detection of MPSK modulations. Equivalently, the SEP of H-S/MRC is lower and upper bounded by

$$P_{e,\mathrm{MRC}}\left(\beta_{\mathrm{L}}^{-1}\Gamma\right) \le P_{e,\text{H-S/MRC}}(\Gamma) \le P_{e,\mathrm{MRC}}\left(\beta_{\mathrm{U}}^{-1}\Gamma\right). \tag{18}$$

Note that β_{L} and β_{U} in (16a) and (16b) do not depend on the average branch SNR and, hence, the SNR penalty bounds in (17) are valid for all values of average branch SNR. Note also that $\beta_{\mathrm{U}} = \beta_{\mathrm{U}}^{\mathrm{A}}$. Using the SEP bounds in (18), the SEP of H-S/MRC at average branch SNR Γ can be lower and upper bounded by the SEP of MRC operating, respectively, at SNRs $\beta_{\mathrm{L}}^{-1}\Gamma$ and $\beta_{\mathrm{U}}^{-1}\Gamma$, using previously published results on MRC. Note that as the difference between β_{L} and $\beta_{\mathrm{L}}^{\mathrm{A}}$ is typically in the second or third significant digit, little is lost by using the rigorous lower bound, to assess the performance of a practical system.

The equivalence of (17) and (18) in *Theorem 2* follows from the definition of SNR penalty in (5), together with the fact that $P_{e,\mathrm{MRC}}(\cdot)$ is a *strict monotonically decreasing* function of its argument. Therefore, in proving *Theorem 2*, it is sufficient to prove either (17) or (18). In Appendix C, we give a proof of the SEP bounds. To do this, we will need some mathematical inequalities, which we derive first in Appendix B.

V. Examples

We now illustrate how the SEP for H-S/MRC can be easily estimated using the results of *Theorem 2*. Fig. 2 shows the exact lower bound and upper bound of the SEP for coherent detection of 4-PSK using H-S/MRC with $(L, N) = (2, 16)$ and $(L, N) = (8, 16)$.[3] The lower and upper bounds are obtained from the SEP of 16-branch MRC operating at SNR $\beta_{\mathrm{L}}^{-1}\Gamma$ and $\beta_{\mathrm{U}}^{-1}\Gamma$, respectively. The exact SNR penalty, $\beta(\Gamma)$, obtained by numerically inverting the curves in Fig. 2, together with β_{L}, $\beta_{\mathrm{L}}^{\mathrm{A}}$ and β_{U} for the case of $(L, N) = (2, 16)$, is plotted as a function of average branch SNR in Fig. 3 and as a function of target SEP

[3]Fig. 2 shows the SEP as low as 10^{-20} only to illustrate the asymptotic behaviors of the $\beta(\Gamma)$; these extremely low SEPs are not practical, especially for wireless mobile communications. Similar comments apply to the ranges of parameters shown in Figs. 1 and 2.

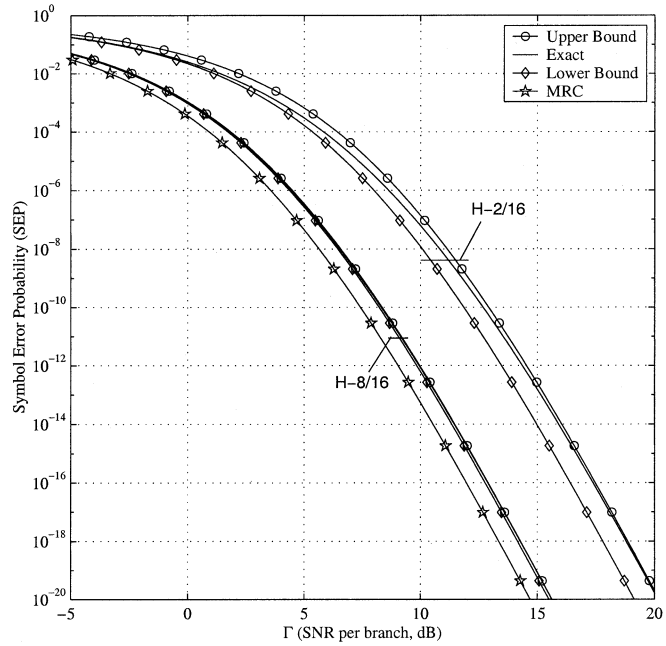

Fig. 2. SEP for coherent detection of 4-PSK with H-S/MRC as a function of average SNR per branch for $N = 16$. The upper and lower bounds are obtained from MRC results according to *Theorem 2*.

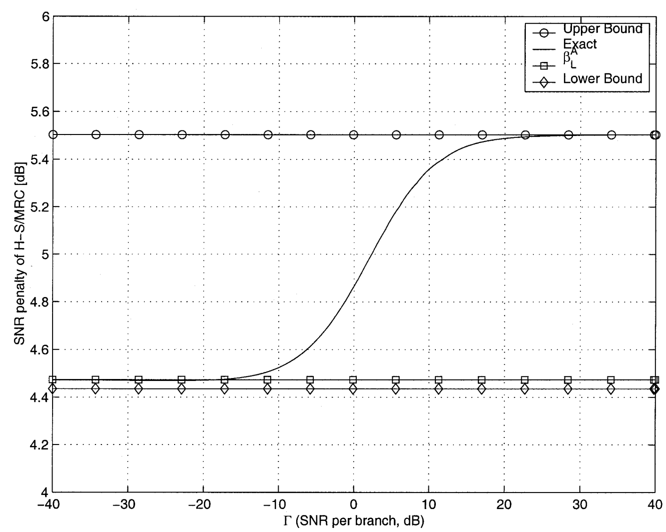

Fig. 3. SNR penalty of H-S/MRC as a function of average SNR per branch for $(L, N) = (2, 16)$.

in Fig. 4. Note again that the SNR penalty is not a constant; it is neither independent of the SNR nor the target SEP.

Figs. 3 and 4 also highlight an interesting result that merits further discussion. Equation (16a) gives β_{L}, a lower bound to the penalty $\beta(\Gamma)$, while (7a) gives the asymptotic, small SNR penalty, $\beta_{\mathrm{L}}^{\mathrm{A}}$. The value β_{L} is also precisely the penalty defined in the SNR sense (i.e, not in a target SEP sense) of H-S/MRC relative to MRC [6], [8]. It is clear from the figures that though $\beta_{\mathrm{L}}^{\mathrm{A}}$ is very close to β_{L}, the two quantities are different, the difference between the two typically being in the second or third significant figure. A test of the validity of this difference has been implemented as follows. A closed-form expression for the SEP of BPSK with MRC is given by [19, p. 825, eq. (14.4–15)] and a closed-form expression for the SEP of SD is found in [11,

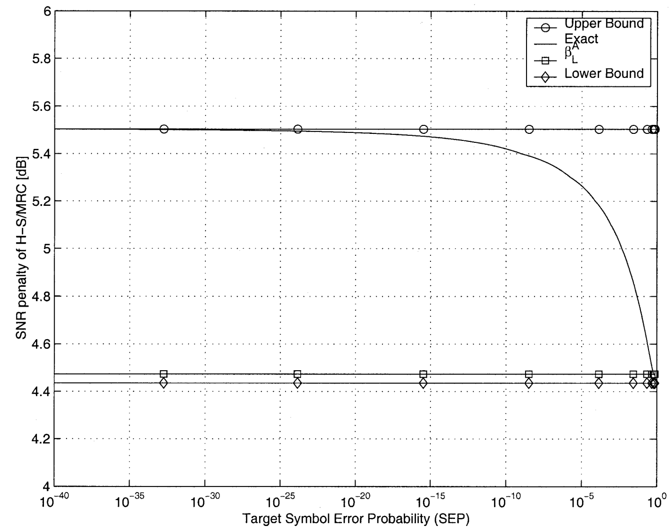

Fig. 4. SNR penalty of H-S/MRC as a function of target SEP for $(L, N) = (2, 16)$.

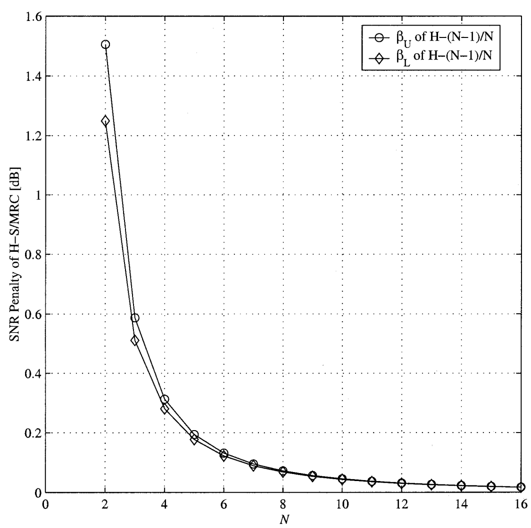

Fig. 5. Penalty incurred by dropping one branch in an H-S/MRC diversity system.

eq. (18)]. Using these, we have calculated the small SNR asymptote for $(L, N) = (1, N), N = 2, 3, \ldots, 8,$ and 16. The results of this test agree with β_L^A as previously determined.

Table I shows β_L and β_L^A for all valid values of $(L, N) \leq 12$. It is clear that β_L provides an excellent approximation to β_L^A in these cases. The values for β_L and β_U can be tabulated using simple formulas given in (16a) and (16b). Table II gives some representative values of the lower and upper bounds on the SNR penalty. The maximum difference between the bounds is less than 0.85 dB for $2 \leq L \leq N \leq 12$. Thus, the geometric mean of the two bounds gives a result that is accurate to within ±0.43 dB for the cases in Table II. As expected, it can be seen from Table II that for a given N, the penalty decreases as L increases. Also as expected, for a given L, the penalty increases as N increases. It is also to be expected that the penalty incurred by dropping one of the diversity branches (i.e., $L = N - 1$) will decrease and become negligible as N increases. This behavior is exhibited in Fig. 5, which quantitatively shows how rapidly

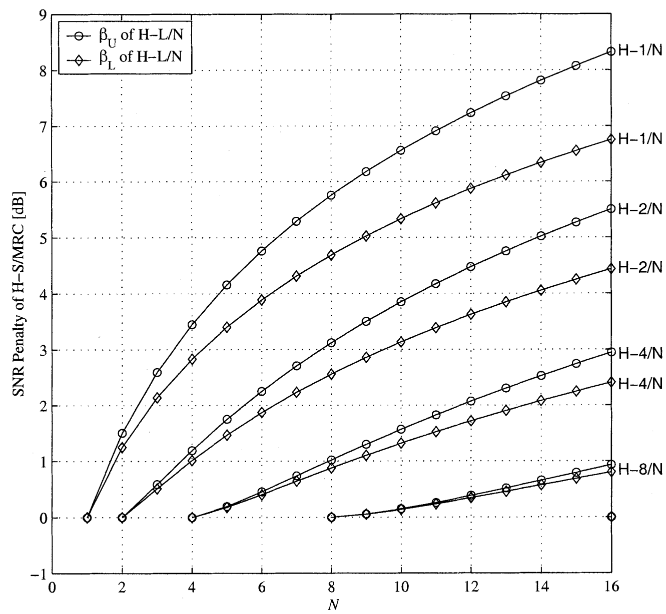

Fig. 6. Lower and upper bounds for the SNR penalty of H-L/N as a function of N for various L.

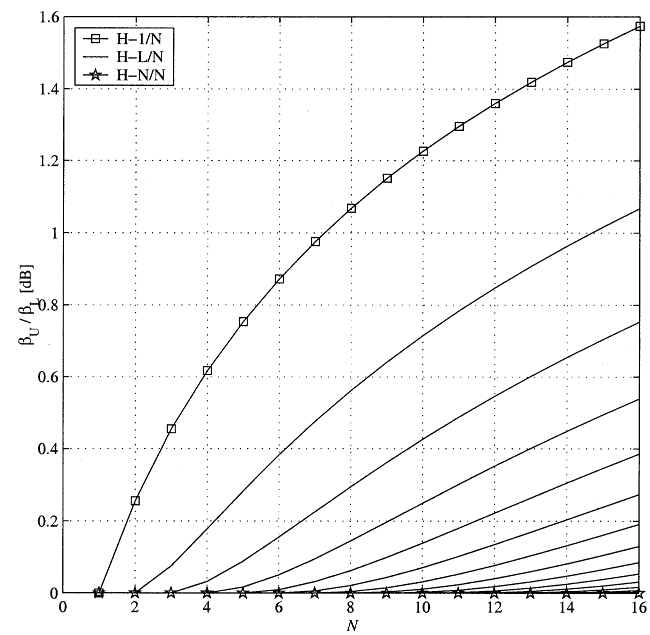

Fig. 7. Ratio β_U / β_L in dB as a function of N for various L. Highest curve is for $L = 1$, and L decreases monotonically to the lowest curve with $L = 16$.

the penalty of H-$(N-1)/N$ decreases as N increases. A knee in the penalty curve occurs around $N = 4$ and the penalty is less than 0.3123 dB for $N \geq 4$.

The SNR penalty of SD relative to MRC is lower and upper bounded by setting $L = 1$ in (16a) and (16b), respectively, to obtain

$$\beta_{L,SD} = \frac{N}{\sum_{n \in \mathbb{Z}_N} \frac{1}{n}} \tag{19a}$$

and

$$\beta_{U,SD} = \{N!\}^{1/N}. \tag{19b}$$

TABLE I
LOWER BOUNDS AND ASYMPTOTIC VALUES $\left\{ \begin{matrix} \beta_{\mathrm{L}} \\ \beta_{\mathrm{L}}^{\mathrm{A}} \end{matrix} \right\}$ OF THE SNR PENALTY IN DECIBELS

	N									
L	3	4	5	6	7	8	9	10	11	12
2	0.5115	1.0146	1.4671	1.8709	2.2333	2.5613	2.8605	3.1355	3.3898	3.6264
	0.5203	1.0312	1.4894	1.8973	2.2628	2.5930	2.8938	3.1701	3.4254	3.6627
3	0	0.2803	0.6048	0.9241	1.2258	1.5077	1.7704	2.0156	2.2451	2.4606
	0	0.2832	0.6116	0.9342	1.2388	1.5230	1.7876	2.0343	2.2650	2.4815
4		0	0.1773	0.4043	0.6420	0.8764	1.1023	1.3179	1.5230	1.7180
		0	0.1786	0.4076	0.6473	0.8837	1.1112	1.3283	1.5347	1.7307
5			0	0.1223	0.2901	0.4741	0.6616	0.8470	1.0274	1.2017
			0	0.1230	0.2919	0.4773	0.6661	0.8526	1.0341	1.2094
6				0	0.0895	0.2187	0.3354	0.5189	0.6737	0.8270
				0	0.0899	0.2198	0.3673	0.5218	0.6775	0.8316
7					0	0.0684	0.1709	0.2906	0.4186	0.5500
					0	0.0686	0.1715	0.2919	0.4206	0.5527
8						0	0.0539	0.1373	0.2368	0.3453
						0	0.0541	0.1377	0.2377	0.3467
9							0	0.0436	0.1127	0.1969
							0	0.0437	0.1130	0.1975
10								0	0.0360	0.0942
								0	0.0361	0.0944
11									0	0.0303
									0	0.0303
12										0
										0

Note that $\beta_{\mathrm{L,SD}}$ is the same as the result given in [10] for the SNR penalty of SD relative to MRC defined in the SNR sense; that is, the degradation in the SNR. This latter penalty measure is appropriate for analog communication systems. The penalty as defined here (the SNR increase required to maintain a target SEP) is appropriate for digital communication systems. Note that the SNR penalty of SD with MPSK in i.i.d. Rayleigh fading channels is lower bounded by (19a) for all values of SNR. Result (19b) is an upper bound to the SNR penalty in digital systems, valid for all values of SNR and is attained at large values of SNR.

Fig. 6 shows β_{L} and β_{U} as functions of the number of diversity branches, N, for various L. It is seen from Fig. 6 that the penalty at large SNR can be significantly underestimated by the lower bound or analog penalty, depending on the values of L and N. This fact can also be observed in Fig. 7, where the ratio $\beta_{\mathrm{U}}/\beta_{\mathrm{L}}$, in decibels, is plotted as a function of N. For example, when $(L, N) = (1, 8)$, the digital penalty at large values of SNR is 1.0683 dB greater than the small (or analog) SNR penalty, and it is 1.5743 dB greater when $(L, N) = (1, 16)$. It is seen in Figs. 6 and 7 that the large SNR penalty becomes increasing larger than (19a) as N increases. In fact

$$\frac{\beta_{\mathrm{U,SD}}}{\beta_{\mathrm{L,SD}}} \approx \frac{\log(N)}{e^{1-1/N}} \tag{20}$$

TABLE II
LOWER AND UPPER BOUNDS $\left\{ \begin{smallmatrix} \beta_{\mathrm{L}} \\ \beta_{\mathrm{U}} \end{smallmatrix} \right\}$ OF THE SNR PENALTY IN DECIBELS

L	N									
	3	4	5	6	7	8	9	10	11	12
2	0.5115	1.0146	1.4671	1.8709	2.2333	2.5613	2.8605	3.1355	3.3898	3.6264
	0.5870	1.1928	1.7501	2.2536	2.7089	3.1229	3.5017	3.8505	4.1735	4.4742
3	0	0.2803	0.6048	0.9241	1.2258	1.5077	1.7704	2.0156	2.2451	2.4606
	0	0.3123	0.6936	1.0797	1.4511	1.8022	2.1321	2.4418	2.7327	3.0067
4		0	0.1773	0.4043	0.6420	0.8764	1.1023	1.3179	1.5230	1.7180
		0	0.1938	0.4550	0.7372	1.0213	1.2992	1.5672	1.8241	2.0697
5			0	0.1223	0.2901	0.4741	0.6616	0.8470	1.0274	1.2017
			0	0.1320	0.3219	0.5368	0.7608	0.9857	1.2074	1.4236
6				0	0.0895	0.2187	0.3654	0.5189	0.6737	0.8270
				0	0.0956	0.2398	0.4089	0.5898	0.7755	0.9617
7					0	0.0684	0.1709	0.2906	0.4186	0.5500
					0	0.0725	0.1857	0.3220	0.4712	0.6270
8						0	0.0539	0.1373	0.2368	0.3453
						0	0.0568	0.1481	0.2603	0.3854
9							0	0.0436	0.1127	0.1969
							0	0.0457	0.1208	0.2149
10								0	0.0360	0.0942
								0	0.0376	0.1005
11									0	0.0303
									0	0.0315
12										0
										0

and the ratio $\beta_{\mathrm{U,SD}}/\beta_{\mathrm{L,SD}}$ grows without bound as N increases. The proof of (20) is a straightforward application of the Stirling formula [20] and is omitted. This interesting result indicates that in digital systems, SD can lose much more in performance relative to MRC than suggested by previous results [10].

VI. CONCLUSIONS

In this paper, we derived simple explicit lower and upper bounds on the SNR penalty of H-S/MRC relative to MRC used with MPSK. The penalty is defined in the error-rate sense as the increase in SNR required for H-S/MRC to achieve the same target SEP as MRC. These bounds are important for the following reason. They are extremely simple and in explicit closed-form, while the exact evaluation of the SEP requires numerical integration. The bounds do not depend on the average branch SNR and, hence, are valid for all values of SNR. Thus, the SEP of H-S/MRC at average branch SNR Γ is lower and upper bounded by the SEP of N-branch MRC operating at SNR $\beta_{\mathrm{L}}^{-1}\Gamma$ and $\beta_{\mathrm{U}}^{-1}\Gamma$, respectively. In the examples, the SEP was approximated to within ± 0.43 dB in SNR for $2 \leq L \leq N \leq 12$.

Contrary to a previous conjecture, the penalty of H-S/MRC diversity relative to MRC diversity was shown not to be a constant; it is neither independent of the SNR nor the target SEP. It was also shown that a previous result for the performance loss of

SD relative to MRC is a lower bound for all values of SNR and can greatly underestimate the loss for large values of SNR. We further obtained an upper bound for the SD performance loss.

APPENDIX A
ASYMPTOTIC EXPANSION OF SEP FOR SMALL SNR

In this appendix, we derive the expansion of SEP for asymptotically small SNR.

Lemma 1: Let

$$p(\Gamma) = \frac{1}{\pi}\int_0^{\Theta}\prod_{n=1}^{N}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+a_n\Gamma}\right)d\theta. \quad (21)$$

The asymptotic expansion of $p(\Gamma)$ for small Γ is given by

$$p(\Gamma) \approx \frac{\Theta}{\pi} - \kappa(a_1,\ldots,a_n)\Gamma^{1/2} + o(\Gamma^{1/2}) \quad (22)$$

where

$$\kappa(a_1,\ldots,a_n) \triangleq \frac{1}{\pi}\int_0^{\infty}\left\{1-\prod_{n=1}^{N}\left(\frac{u^2}{a_n+u^2}\right)\right\}du. \quad (23)$$

Proof: Let

$$g(\Gamma) = \frac{\Theta}{\pi} - p(\Gamma). \quad (24)$$

For any $\epsilon \geq 0$, let $\tilde{\epsilon} = \epsilon/K(a_1,\ldots,a_n)$. The continuity of $\theta/\sin(\theta)$ around $\theta = 0$, implies that there exists $\delta(\epsilon)$ such that

$$\frac{1}{1+\tilde{\epsilon}} \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{1-\tilde{\epsilon}} \quad (25)$$

whenever $|\theta| \leq \delta(\epsilon)$. For such $\delta(\epsilon)$, $g(\Gamma)$ can be rewritten in terms of two separate integrals as

$$g(\Gamma) = I_1(\Gamma,\epsilon) + I_2(\Gamma,\epsilon) \quad (26)$$

where

$$I_1(\Gamma,\epsilon) \triangleq \frac{1}{\pi}\int_0^{\delta(\epsilon)}\left\{1-\prod_{n=1}^{N}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+a_n\Gamma}\right)\right\}d\theta \quad (27a)$$

and

$$I_2(\Gamma,\epsilon) \triangleq \frac{1}{\pi}\int_{\delta(\epsilon)}^{\Theta}\left\{1-\prod_{n=1}^{N}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+a_n\Gamma}\right)\right\}d\theta. \quad (27b)$$

We will consider $I_1(\Gamma,\epsilon)$ and $I_2(\Gamma,\epsilon)$ separately in the following.

We will first show that

$$I_2(\Gamma,\epsilon) \approx o(\Gamma^{1/2}). \quad (28)$$

Let $A = \max_n a_n$, then

$$\begin{aligned} I_2(\Gamma,\epsilon) &\leq \frac{1}{\pi}\int_{\delta(\epsilon)}^{\Theta}\left\{1-\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+A\Gamma}\right)^N\right\}d\theta \\ &= \frac{1}{\pi}\int_{\delta(\epsilon)}^{\Theta}\left\{1-\left(1-\frac{A\Gamma}{A\Gamma+\sin^2(\theta)}\right)^N\right\}d\theta. \end{aligned} \quad (29)$$

It can be shown by induction that $1-(1-q)^N \leq Nq, \forall N \geq 1$ and $q \leq 1$. Using this fact, (29) becomes

$$I_2(\Gamma,\epsilon) \leq \frac{1}{\pi}\int_{\delta(\epsilon)}^{\Theta}\frac{NA\Gamma}{A\Gamma+\sin^2(\theta)}d\theta. \quad (30)$$

Letting $\tilde{\delta}(\epsilon,M) = \min\{\delta(\epsilon),\pi(M-1)/M\}$, (30) can be upper bounded as

$$I_2(\Gamma,\epsilon) \leq \frac{1}{\pi}\int_{\delta(\epsilon)}^{\Theta}\frac{NA\Gamma}{A\Gamma+\sin^2(\tilde{\delta}(\epsilon,M))}d\theta.$$

This implies that

$$\frac{1}{\Gamma^{1/2}}I_2(\Gamma,\epsilon) \leq \frac{1}{\pi}\frac{NA[\Theta-\delta(\epsilon)]}{A\Gamma+\sin^2(\tilde{\delta}(\epsilon,M))}\Gamma^{1/2} \quad (31)$$

and hence

$$\limsup_{\Gamma\to 0}\frac{1}{\Gamma^{1/2}}I_2(\Gamma,\epsilon) \leq 0. \quad (32)$$

On the other hand, it is clear from the definition of I_2 in (27b) that $I_2(\Gamma,\epsilon) \geq 0$. This together with (32) gives

$$\lim_{\Gamma\to 0}\frac{1}{\Gamma^{1/2}}I_2(\Gamma,\epsilon) = 0. \quad (33)$$

This completes the proof of (28).

Next, we consider $I_1(\Gamma,\epsilon)$. We will show that for any $\epsilon \geq 0$

$$\limsup_{\Gamma\to 0}\left|\frac{1}{\Gamma^{1/2}}I_1(\Gamma,\epsilon) - \kappa(a_1,\ldots,a_n)\right| \leq \epsilon. \quad (34)$$

From (27a)

$$I_1(\Gamma,\epsilon) = \frac{1}{\pi}\int_0^{\delta(\epsilon)}\left\{1-\prod_{n=1}^{N}\left(1-\frac{a_n\Gamma}{a_n\Gamma+\sin^2(\theta)\frac{\theta^2}{\theta^2}}\right)\right\}d\theta. \quad (35)$$

Using (25), (35) can be upper bounded as

$$\begin{aligned} I_1(\Gamma,\epsilon) &\leq \frac{1}{\pi}\int_0^{\delta(\epsilon)}\left\{1-\prod_{n=1}^{N}\left(1-\frac{a_n\Gamma}{a_n\Gamma+\frac{\theta^2}{(1+\tilde{\epsilon})^2}}\right)\right\}d\theta \\ &= \frac{1}{\pi}\int_0^{(\delta(\epsilon)/(1+\tilde{\epsilon}))(1/\Gamma^{1/2})}\left\{1-\prod_{n=1}^{N}\left(\frac{u^2}{a_n+u^2}\right)\right\} \\ &\quad \cdot \Gamma^{1/2}(1+\tilde{\epsilon})du \end{aligned} \quad (36)$$

where we have obtained (36) by the change of variables $u = (1/\Gamma^{1/2})(\theta/(1+\tilde{\epsilon}))$. Taking the $\limsup$ on both sides of (36) gives

$$\limsup_{\Gamma\to 0}\frac{1}{\Gamma^{1/2}}I_1(\Gamma,\epsilon) \leq \kappa(a_1,\ldots,a_n)(1+\tilde{\epsilon}) \quad (37)$$

and therefore

$$\limsup_{\Gamma\to 0}\left[\frac{1}{\Gamma^{1/2}},I_1(\Gamma,\epsilon) - \kappa(a_1,\ldots,a_n)\right] \leq \epsilon. \quad (38)$$

Similar steps to those leading to (38) yield

$$\liminf_{\Gamma\to 0}\left[\frac{1}{\Gamma^{1/2}}I_1(\Gamma,\epsilon) - \kappa(a_1,\ldots,a_n)\right] \geq -\epsilon. \quad (39)$$

Equations (38) and (39) imply that

$$\limsup_{\Gamma\to 0}\left|\frac{1}{\Gamma^{1/2}}I_1(\Gamma,\epsilon) - \kappa(a_1,\ldots,a_n)\right| \leq \epsilon \quad (40)$$

which completes the proof of (34).

Recall from (26) that $I_1(\Gamma,\epsilon)=g(\Gamma)-I_2(\Gamma,\epsilon)$. Substituting this into (40), and using (28) results in

$$\limsup_{\Gamma\to 0}\left|\frac{1}{\Gamma^{1/2}}g(\Gamma)-\kappa(a_1,\ldots,a_n)\right|\le\epsilon. \tag{41}$$

The above is true for all $\epsilon\ge 0$, and thus

$$\lim_{\Gamma\to 0}\frac{1}{\Gamma^{1/2}}g(\Gamma)=\kappa(a_1,\ldots,a_n). \tag{42}$$

This, together with (24), implies that

$$p(\Gamma)=\frac{\Theta}{\pi}-\kappa(a_1,\ldots,a_n)\Gamma^{1/2}+o(\Gamma^{1/2}) \tag{43}$$

which completes the proof of *Lemma 1*. □

APPENDIX B
MATHEMATICAL INEQUALITIES

In this appendix, we derive some mathematical inequalities needed to prove the SEP bounds in Appendix C. Let $\boldsymbol{x}=\{x_n\}_{n\in\mathbb{Z}_N}$ be a vector whose elements are N nonnegative numbers and $\boldsymbol{p}=\{p_n\}_{n\in\mathbb{Z}_N}$ be a probability vector associated with $\boldsymbol{x}$ such that $\mathbb{P}\{x_n\}=p_n$ and $\sum_{n\in\mathbb{Z}_N}p_n=1$.

Definition 1: As in [21], we define the arithmetic and geometric $\boldsymbol{p}$-mean (AGM) to be

$$\mathfrak{A}(\boldsymbol{x},\boldsymbol{p})\triangleq\sum_{n\in\mathbb{Z}_N}p_nx_n$$

and

$$\mathfrak{G}(\boldsymbol{x},\boldsymbol{p})\triangleq\prod_{n\in\mathbb{Z}_N}x_n^{p_n}$$

respectively.

Theorem 3 (AGM Inequality): The arithmetic and geometric $\boldsymbol{p}$-mean satisfy the following relation:

$$\mathfrak{A}(\boldsymbol{x},\boldsymbol{p})\ge\mathfrak{G}(\boldsymbol{x},\boldsymbol{p}) \tag{44}$$

and the *equality* in (44) is achieved *if and only if* (iff) $x_n=x$ for all n satisfying $p_n>0$. Several proofs of *Theorem 3* are given in [21, pp. 16–18].

Definition 2: Let $\boldsymbol{y}=\{y_n\}_{n\in\mathbb{Z}_N}$. The jth *elementary symmetric function* (ESF) of $\boldsymbol{y}$, denoted by $\mathfrak{E}_j(\boldsymbol{y})$, is defined as the sum of all possible products (j at a time) of the elements of $\boldsymbol{y}$. Mathematically

$$\mathfrak{E}_j(\boldsymbol{y})\triangleq\sum_{S\in\mathcal{S}_j}\prod_{n\in S}y_n \tag{45}$$

where $\mathcal{S}_j=\{S\subset\mathbb{Z}_N:|S|=j\}$ and $|S|$ denotes the cardinality of the set S.

Theorem 4 (ESF-Sum Inequality): If the elements of $\boldsymbol{y}$ are nonnegative, then the sum of the ESF's satisfy the following inequality:

$$\sum_{j\in\mathbb{Z}_N}\mathfrak{E}_j(\boldsymbol{y})\ge\sum_{j\in\mathbb{Z}_N}\binom{N}{j}(y_1y_2\ldots y_N)^{j/N} \tag{46}$$

and the *equality* in (46) is achieved iff all elements of $\boldsymbol{y}$ are equal.

Proof: For each $S\in\mathcal{S}_j$, let

$$x(S)=\prod_{n\in S}y_n.$$

Since the elements of $\boldsymbol{y}$ are nonnegative, $\{x(S)\}_{S\in\mathcal{S}_j}$ are $|\mathcal{S}_j|$ nonnegative numbers, and from *Theorem 3*

$$\sum_{S\in\mathcal{S}_j}p(S)x(S)\ge\prod_{S\in\mathcal{S}_j}[x(S)]^{p(S)} \tag{47}$$

for any probability vector $\{p(S)\}_{S\in\mathcal{S}_j}$ such that $\sum_{S\in\mathcal{S}_j}p(S)=1$. In particular, with $p(S)=1/|\mathcal{S}_j|$, (47) becomes

$$\frac{1}{|\mathcal{S}_j|}\sum_{S\in\mathcal{S}_j}x(S)\ge\left[\prod_{S\in\mathcal{S}_j}x(S)\right]^{1/|\mathcal{S}_j|}. \tag{48}$$

Consider now the following product:

$$\prod_{S\in\mathcal{S}_j}x(S)=\prod_{S\in\mathcal{S}_j}\prod_{n\in S}y_n. \tag{49}$$

Note that the first product on the right side (RS) of (49) has $|\mathcal{S}_j|=\binom{N}{j}$ terms and the second product has j terms. Out of $|\mathcal{S}_j|$ terms, the number of terms in which y_1 occurs is equal to $\binom{N-1}{j-1}$. By symmetry, similar arguments show that y_n occurs $\binom{N-1}{j-1}$ times in the RS of (49) for each $n\in\mathbb{Z}_N$. Therefore, (49) becomes

$$\prod_{S\in\mathcal{S}_j}x(S)=\prod_{n\in\mathbb{Z}_N}y_n^{\binom{N-1}{j-1}}. \tag{50}$$

Note that

$$\binom{N}{j}j=N\binom{N-1}{j-1} \tag{51}$$

and therefore

$$\frac{j}{N}=\frac{\binom{N-1}{j-1}}{|\mathcal{S}_j|}. \tag{52}$$

Substituting (50) and (52) into (48) gives

$$\frac{1}{|\mathcal{S}_j|}\sum_{S\in\mathcal{S}_j}x(S)\ge(y_1y_2,\ldots,y_N)^{j/N}. \tag{53}$$

Multiplying both sides by $|\mathcal{S}_j|$ and summing over j, (53) becomes

$$\sum_{j\in\mathbb{Z}_N}\sum_{S\in\mathcal{S}_j}x(S)\ge\sum_{j\in\mathbb{Z}_N}\binom{N}{j}(y_1y_2,\ldots,y_N)^{j/N}. \tag{54}$$

But $\sum_{S\in\mathcal{S}_j}x(S)=\sum_{S\in\mathcal{S}_j}\prod_{n\in S}y_n=\mathfrak{E}_j(\boldsymbol{y})$, and therefore

$$\sum_{j\in\mathbb{Z}_N}\mathfrak{E}_j(\boldsymbol{y})\ge\sum_{j\in\mathbb{Z}_N}\binom{N}{j}(y_1y_2,\ldots,y_N)^{j/N}.$$

Note that for each j, *Theorem 3* implies equality (48) iff $x(S)=x\,\forall S\in\mathcal{S}_j$. But $x(S)=x\,\forall S\in\mathcal{S}_j$ iff $y_n=y\,\forall n\in\mathbb{Z}_N$, which implies that for each j, the equality in (48), and consequently, in (53), is achieved iff $y_n=y\;\forall n\in\mathbb{Z}_N$. Since the equality holds for each j, summing over j preserves the equality in (54), and the equality in the ESF-Sum Inequality is achieved iff all elements of $\boldsymbol{y}$ are equal. This completes the proof of *Theorem 4*. □

APPENDIX C
PROOF OF THE SEP BOUNDS

In this appendix, we give a proof of the SEP bounds using the results of Appendix B. In particular, we will use the AGM Inequality, given by (44) of *Theorem 3*, to prove the lower bound. Similarly, the ESF-Sum Inequality, given by (46) of *Theorem 4*, will be used to prove the upper bound.

Proof [Lower Bound]: For each Γ and θ, let

$$x_n = \begin{cases} \dfrac{\Gamma + \sin^2\theta}{\sin^2\theta}, & n \in \mathbb{Z}_L \\ \dfrac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}, & n \in \mathbb{Z}_L^N. \end{cases} \tag{55}$$

Since $x_n \geq 0$, *Theorem 3* implies that, for any probability vector $\boldsymbol{p}$

$$\sum_{n\in\mathbb{Z}_L} p_n \left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right] + \sum_{n\in\mathbb{Z}_L^N} p_n \left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right] \geq \prod_{n\in\mathbb{Z}_L}\left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right]^{p_n} \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right]^{p_n} \tag{56}$$

$$\left[\frac{\Gamma\left(\sum_{n\in\mathbb{Z}_L} p_n + L\sum_{n\in\mathbb{Z}_L^N} p_n \frac{1}{n}\right) + \sin^2\theta}{\sin^2\theta}\right] \geq \prod_{n\in\mathbb{Z}_L}\left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right]^{p_n} \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right]^{p_n}. \tag{57}$$

For N i.i.d. diversity branches, $\boldsymbol{p}$ is a $N \times 1$ vector with identical elements $p_n = 1/N$, $n \in \mathbb{Z}_N$ and $[\sum_{n\in\mathbb{Z}_L} p_n + L\sum_{n\in\mathbb{Z}_L^N} p_n(1/n)] = {\beta_L}^{-1}$ in accordance with (16a), and therefore[4]

$$\left[\frac{\beta_{\mathrm{L}}^{-1}\Gamma + \sin^2\theta}{\sin^2\theta}\right]^N \geq \left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right]^L \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right]. \tag{58}$$

Integrating the inverse of both sides over θ and scaling by $1/\pi$, we obtain

$$\frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{\beta_{\mathrm{L}}^{-1}\Gamma + \sin^2\theta}\right]^N d\theta \leq \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{\Gamma + \sin^2\theta}\right]^L \cdot \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\sin^2\theta}{\Gamma L/n + \sin^2\theta}\right] d\theta. \tag{59}$$

Applying (59) with Γ replaced by $c_{\mathrm{MPSK}}\Gamma$ and comparing it with (3) and (4), we obtain the lower bound for the SEP of H-S/MRC in terms of the well-known MRC performance for each Γ as

$$P_{e,\mathrm{MRC}}(\beta_{\mathrm{L}}^{-1}\Gamma) \leq P_{e,\text{H-S/MRC}}(\Gamma). \tag{60}$$

□

Proof [Upper Bound]: The ESF-Sum Inequality of *Theorem 4* is equivalent to

$$1 + \sum_{j\in\mathbb{Z}_N} \mathfrak{E}_j(\boldsymbol{y}) \geq 1 + \sum_{j\in\mathbb{Z}_N}\binom{N}{j}\left(\prod_{n\in\mathbb{Z}_N} y_n^{1/N}\right)^j. \tag{61}$$

Note that the LS is the expansion of the N-product of $(y_n + 1)$ and the RS is the binomial expansion of $[\prod_{n\in\mathbb{Z}_N} y_n^{1/N} + 1]^N$. Therefore

$$\prod_{n\in\mathbb{Z}_N}(y_n + 1) \geq \left[\prod_{n\in\mathbb{Z}_N} y_n^{1/N} + 1\right]^N. \tag{62}$$

For each Γ and θ, let

$$y_n = \begin{cases} \dfrac{\Gamma}{\sin^2\theta}, & n \in \mathbb{Z}_L \\ \dfrac{\Gamma L/n}{\sin^2\theta}, & n \in \mathbb{Z}_L^N. \end{cases} \tag{63}$$

Since $y_n \geq 0\ \forall n \in \mathbb{Z}_N$, then (62) becomes

$$\left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right]^L \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right] \geq \left[\frac{\Gamma\prod_{n\in\mathbb{Z}_L^N}\left(\frac{L}{n}\right)^{1/N} + \sin^2\theta}{\sin^2\theta}\right]^N. \tag{64}$$

But $[\prod_{n\in\mathbb{Z}_L^N} L/n]^{1/N} = \beta_{\mathrm{U}}^{-1}$, and (64) becomes

$$\left[\frac{\Gamma + \sin^2\theta}{\sin^2\theta}\right]^L \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\Gamma L/n + \sin^2\theta}{\sin^2\theta}\right] \geq \left[\frac{\beta_{\mathrm{U}}^{-1}\Gamma + \sin^2\theta}{\sin^2\theta}\right]^N. \tag{65}$$

Therefore, for each Γ we have

$$\frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{\Gamma + \sin^2\theta}\right]^L \prod_{n\in\mathbb{Z}_L^N}\left[\frac{\sin^2\theta}{\Gamma L/n + \sin^2\theta}\right] d\theta \leq \frac{1}{\pi}\int_0^{\Theta}\left[\frac{\sin^2\theta}{\beta_{\mathrm{U}}^{-1}\Gamma + \sin^2\theta}\right]^N d\theta \tag{66}$$

and hence, applying (66) with Γ replaced by $c_{\mathrm{MPSK}}\Gamma$, we obtain

$$P_{e,\text{H-S/MRC}}(\Gamma) \leq P_{e,\mathrm{MRC}}(\beta_{\mathrm{U}}^{-1}\Gamma). \tag{67}$$

[4]Note that (58) can also be thought of as a consequence of Schur monotonicity [22].

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Moe Z. Win (S'85–M'87–SM'97) received the B.S. degree (magna cum laude) from Texas A&M University, College Station, and the M.S. degree from the University of Southern California (USC), Los Angeles, in 1987 and 1989, respectively, in electrical engineering. As a Presidential Fellow at USC, he received both an M.S. degree in applied mathematics and the Ph.D. degree in electrical engineering in 1998. He is a Distinguished Alumnus of Mountain View College.

In 1987, he joined the Jet Propulsion Laboratory (JPL), California Institute of Technology, Pasadena, where he performed research on digital communications and optical systems for NASA space exploration missions. From 1994 to 1997, he was a Research Assistant with the Communication Sciences Institute, USC, where he played a key role in the successful creation of the Ultra-Wideband Radio Laboratory. From 1998 to 2002, he was with the Wireless Systems Research Department, AT&T Laboratories-Research, Middletown, NJ. Since 2002, he has been with the Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, where he holds the Charles Stark Draper Chair. His main research interests are the application of mathematical and statistical theories to communication, detection, and estimation problems including measurement and modeling of time-varying channels, design and analysis of multiple antenna systems, ultra-wide bandwidth (UWB) communications systems, optical communications systems, and space communications systems.

Dr. Win has been actively involved in organizing and chairing sessions, and has served as a member of the Technical Program Committee in a number of international conferences. He currently serves as the Technical Program Chair for the IEEE Communication Theory Symposium of ICC 2004. He served as the Technical Program Chair for the IEEE Communication Theory Symposium of Globecom 2000 and the IEEE Conference on Ultra Wideband Systems and Technologies (2002), Technical Program Vice-Chair for the IEEE International Conference on Communications (2002), and the Tutorial Chair for the IEEE Semiannual International Vehicular Technology Conference (Fall 2001). He is the secretary for the Radio Communications Technical Committee, the current Editor for Equalization and Diversity for the IEEE TRANSACTIONS ON COMMUNICATIONS and a Guest Editor for the 2002 IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, Special Issue on Ultra -Wideband Radio in Multiaccess Wireless Communications. He received the IEEE Communications Society Best Student Paper Award at the Fourth Annual IEEE NetWorld+Interop '97 Conference in 1997, International Telecommunications Innovation Award from the Korea Electronics Technology Institute in 2002, Young Investigator Award from the Office of Naval Research in 2003, and IEEE Antennas and Propagation Society S. A. Schelkunoff Transactions Prize Paper Award in 2003.

Norman C. Beaulieu (S'82–M'86–SM'89–F'99) received the B.A.Sc. (honors), M.A.Sc., and Ph.D. degrees in electrical engineering from the University of British Columbia, Vancouver, BC, Canada, in 1980, 1983, and 1986, respectively.

He was a Queen's National Scholar Assistant Professor with the Department of Electrical Engineering, Queen's University, Kingston, ON, Canada, from September 1986 to June 1988, an Associate Professor from July 1988 to June 1993, and a Professor from July 1993 to August 2000. In September 2000, he became the *i*CORE Research Chair in Broadband Wireless Communications at the University of Alberta, Edmonton, AB, Canada, and in January 2001, the Canada Research Chair in Broadband Wireless Communications. His current research interests include broadband digital communications systems, fading channel modeling and simulation, interference prediction and cancellation, and decision-feedback equalization.

Dr. Beaulieu is a Member of the IEEE Communication Theory Committee and served as its Representative to the Technical Program Committee of the 1991 International Conference on Communications and as Co-Representative to the Technical Program Committee of the 1993 International Conference on Communications and the 1996 International Conference on Communications. He was General Chair of the Sixth Communication Theory Mini-Conference in association with GLOBECOM 97 and Co-Chair of the Canadian Workshop on Information Theory 1999. He has been an Editor for Wireless Communication Theory of the IEEE TRANSACTIONS ON COMMUNICATIONS since January 1992, an Associate Editor for Wireless Communication Theory of the *IEEE Communications Letters* since November 1996, Editor-in-Chief of the IEEE TRANSACTIONS ON COMMUNICATIONS since January 2000, and on the Editorial Board of the *Proceedings of the IEEE* since November 2000. He received the Natural Science and Engineering Research Council of Canada (NSERC) E. W. R. Steacie Memorial Fellowship in 1999. He was awarded the University of British Columbia Special University Prize in Applied Science in 1980 as the highest standing graduate in the faculty of Applied Science. He is a Fellow of The Royal Society of Canada.

Lawrence A. Shepp received the B.S. degree in applied mathematics from the Polytechnic Institute of Brooklyn, Brooklyn, NY, in 1958, and the M.A. and Ph.D. degrees in mathematics from Princeton University, Princeton, NJ, in 1960 and 1961, respectively.

He joined the Mathematics Research Center of AT&T Bell Laboratories in 1962 and became a Distinguished Member of Technical Staff at Bell Laboratories in 1986. He was a Professor of Statistics and Operations Research at Columbia University, New York, NY, from 1996 to 1997. In 1997, he joined the Department of Statistics at Rutgers University, Piscataway, NJ, where he is Professor. During his career at AT&T, he held various joint appointments: he was a Professor of Radiology at Columbia University from 1973 to 1996, a Mathematician in Radiology Service at the Columbia Presbyterian Hospital from 1974 to 1996, and a Professor of Statistics (1/4 time) at Stanford University, Stanford, CA, from 1978 to 1992. He was a member of the Scientific Boards of American Science and Engineering Inc. from 1974 to 1975, and of Resonex Inc. from 1983 to 1984. He has a permanent visiting position at University of Coral Gables, Coral Gables, FL, and also at Stanford University. His current research interests include functional magnetic resonance imaging, tomography, wireless telephony, cosmology, and the mathematics of finance.

Dr. Shepp has made fundamental contributions to CAT scanning, emission tomography, probability theory (random covering, Gaussian processes, connectedness of random graphs), mathematical finance, and economics. He is a Co-Editor for the *Wiley International Journal of Imaging Systems and Technology* and was an Associate Editor for *Journal of Computer Assisted Tomography* from 1977 to 1992. He was elected to the National Academy of Science, the Institute of Medicine, and the Academy of Arts and Science. He was the winner of the William Lowell Putnam Intercollegiate Mathematics Competition in 1958, the Paul Levy Prize in 1966, and he received an IEEE Distinguished Scientist Award in 1979.

Benjamin F. Logan, Jr. received the B.S. degree from Texas Technological College, Lubbock, in 1946, the M.S. degree from Massachusetts Institute of Technology (MIT), Cambridge, in 1951, and the Eng.Sc.D. degree from Columbia University, New York, in 1965, all in electrical engineering.

While at MIT, he was a Research Assistant in the Research Laboratory of Electronics, investigating characteristics of high-power electrical discharge lamps with Dr. Harold E. Edgerton. Also at MIT, he was a member of the Dynamic Analysis and Control Laboratory, where he was engaged in analog computer development. From 1955 to 1956 he was with Hycon-Eastern, Inc., where he was concerned with the design of airborne power supplies. From 1956 to 1993, he was with AT&T Bell Laboratories. He first joined the Visual and Acoustic Research Department in July 1956, where he conducted research in the processing of speech signals. Then in 1962, he joined the Mathematical Research Center of AT&T Bell Laboratories, where he studied and established interesting properties of high-pass signals.

Dr. Logan has made fundamental contributions to echo cancellation in telephony which led to early key patents, tomography, and a Fourier transform theory of band-limited and band-pass functions. He was elected to the Academy of Electrical Engineering by Texas Tech University in 1997, for outstanding achievements at Bell Laboratories, where he was a Distinguished Member of Technical Staff. He presented an invited paper on "The Properties of High-Pass Signals" at the American Mathematical Society meeting, Hartford, CT, in 1995. He continues his interest in and his study of these subjects after his retirement from AT&T Bell Laboratories in 1993.

Jack H. Winters (S'77–M'81–SM'88–F'96) received the B.S.E.E. degree from the University of Cincinnati, Cincinnati, OH, in 1977, and the M.S. and Ph.D. degrees in electrical engineering from The Ohio State University, Columbus, in 1978 and 1981, respectively.

From 1981 to early 2002, he was with AT&T Bell Laboratories, and then AT&T Labs—Research, Middletown, NJ, where he was Division Manager of the Wireless Systems Research Department. Since early 2002, he has been consulting for several wireless and optical communication companies and is Chief Scientist at Motia, Inc., Middletown, NJ. He has studied signal processing techniques for increasing the capacity and reducing signal distortion in fiber optic, mobile radio, and indoor radio systems, and is currently studying smart antennas, adaptive arrays, and equalization for indoor and mobile radio systems.

Dr. Winters is an IEEE Distinguished Lecturer for both the IEEE Communications and Vehicular Technology Societies, Area Editor for Transmission Systems for the IEEE TRANSACTIONS ON COMMUNICATIONS, and New Jersey Inventor of the Year for 2001.